Polynomial Representations of Symplectic Groups

by

Maria Iano-Fletcher

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Introduction.

In Chapter 1 we reveiw some of the classical theory of reductive algebraic groups over an algebraically closed field.

In Chapter 2 we summarise the work of Green [1] on polynomial representations of $GL_n(\mathbf{k})$ where $\mathbf{k}$ is an infinite field. The irreducible polynomial reprentations of degree r of $GL_n(\mathbf{k})$ are parametrised by partitions λ of r. The irreducible polynomial $GL_n(\mathbf{k})$-module $F_{\lambda,\mathbf{k}}$ can be obtained in two ways, as the quotient of a Weyl module $V_{\lambda,\mathbf{k}}$ by its unique maximal submodule, or as the unique minimal submodule of a Shur module $D_{\lambda,\mathbf{k}}$. The two modules $V_{\lambda,\mathbf{k}}$ and $D_{\lambda,\mathbf{k}}$ are duals of one another. We describe Green's basis of $D_{\lambda,\mathbf{k}}$ consisting of one element D_T for each semistandard λ-tableau T. We also describe the results of Carter and Lusztig on Weyl modules and the Carter-Lusztig basis of $V_{\lambda,\mathbf{k}}$ consisting of one element ψ_T for each semistandard λ-tableau T.

The aim of this thesis is to extend Green's work to polynomial representations of symplectic groups over an infinite field. The basis facts about symplectic groups are described in Chapter 3.

In Chapter 4 we introduce the idea of symplectic tableaux due to R. C. King. The dimensions of the Weyl module $V^{sp}_{\lambda,\mathbf{k}}$ and the Shur module $D^{sp}_{\lambda,\mathbf{k}}$ is equal to the number of symplectic λ-tableaux. We shall use symplectic tableaux to parametrise basis vectors of the Weyl and Shur modules.

Chapter 5 is a detailed study of the Weyl module $V^{sp}_{\lambda,\mathbf{k}}$. We define an element $V_T \in V^{sp}_{\lambda,\mathbf{k}}$ for each symplectic λ-tableau T, and we show that the V_T form a basis for $V^{sp}_{\lambda,\mathbf{k}}$. To do so we use contraction operators. The simple contraction operators were introduced by Weyl. More general contraction operators are necessary to make it work over an arbitary infinite field. The elements of $V^{sp}_{\lambda,\mathbf{k}}$ are all traceless, that is, they are in the kernal of all contraction operators.

In Chapter 6 we consider the Shur module $D^{sp}_{\lambda,\mathbf{k}}$. We describe an element $D_T \in D^{sp}_{\lambda,\mathbf{k}}$ for each symplectic λ-tableau T, and show that the D_T form a basis for $D^{sp}_{\lambda,\mathbf{k}}$. To show that these elements span the module we must use expansion operators. Again the simple expansion operators are not sufficient, but we show they are sufficient in characteristic zero or when $\operatorname{char}(\mathbf{k}) > \frac{1}{2}\mu_1$, where μ is the conjugate partition to λ. To sho the linear independence of the elements D_T we use a proof similar in style to a proof of Green for $GL_n(\mathbf{k})$. Finally we show $D^{sp}_{\lambda,\mathbf{k}} \cong I^{sp}_{\lambda,\mathbf{k}}$, where $I^{sp}_{\lambda,\mathbf{k}} = \mathrm{H}^0(G/B, \lambda)$, the induced module.

In Chapter 7 we show that $V^{sp}_{\lambda,\mathbf{k}}$ and $D^{sp}_{\lambda,\mathbf{k}}$ are dual modules. We define an invariant form mapping $V^{sp}_{\lambda,\mathbf{k}} \times D^{sp}_{\lambda,\mathbf{k}}$ to $\mathbf{k}$. We show there is an involution δ on the set of symplectic λ-tableaux such that $(v_T, D_{\delta(T)}) = \pm 1$ and $(v_{T'}, D_{\delta(T)}) = 0$ unless $T' \geq T$ with respect to a suitable ordering on the set of symplectic λ-tableaux. Finally we mention other approaches to the same subject.

The first four chapters of this thesis are expositary, and the final three are original.

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1

Reductive Algebraic Groups.

We are interested in symplectic groups over any infinite field, and shall begin with the case when the field is algebraically closed. Over an algebraically closed field symplectic groups are simple algebraic groups. The theory of algebraic groups over an algebraically closed field has been studied a great deal, and many classical results are known. We briefly review some of the classical theory here, with the aim of extending some of it to the more general case over any infinite field. We shall state the relevant results without proof. The sources for this information are Borel [1], Borel et al [1], Carter [1] and [2] and Humphreys [1] and [2]. Throughout this chapter $\mathbf{K}$ is an algebraically closed field.

1.1 Definition of an Affine Algebraic Group.

An affine algebraic group is both a group and an affine variety. We begin by discussing the basic properties of affine varieties.

Let $\mathbf{K}$ be an algebraically closed field and let $\mathbf{K}^n$ be the vector space of n-tuples over $\mathbf{K}$. The polynomial ring $\mathbf{K}[x_1, \ldots, x_n]$ gives rise to a ring of functions from $\mathbf{K}^n$ to $\mathbf{K}$. Let S be a subset of $\mathbf{K}[x_1, \ldots, x_n]$. We define $\mathcal{V}(S) \subset \mathbf{K}^n$ by

$$\mathcal{V}(S) = \{v \in \mathbf{K}^n; f(v) = 0 \text{ for all } f \in S\}.$$

Such a subset of $\mathbf{K}^n$ is called an *affine variety,* and satisfies

$$\mathcal{V}(S) = \mathcal{V}(I)$$

where I is the ideal in $\mathbf{K}[x_1, \ldots, x_n]$ generated by S.

Given an affine variety $V \subset \mathbf{K}[x_1, \ldots, x_n]$ its ideal $\mathcal{I}(V)$ is defined to be

$$\mathcal{I}(V) = \{f \in \mathbf{K}[x_1, \ldots, x_n]; f(v) = 0 \text{ for all } v \in V\}.$$

Thus every ideal I determines an affine variety $\mathcal{V}(I)$ and every affine variety V determines an ideal $\mathcal{I}(V)$. The relation between the operations $\mathcal{V}$ and $\mathcal{I}$ is as follows. For any affine variety V

$$\mathcal{V}(\mathcal{I}(V)) = V.$$

However it is not true that $\mathcal{I}(\mathcal{V}(I)) = I$ for an ideal I. Instead for every ideal I

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I},$$

where $\sqrt{I}$ is the radical of I.

It follows that an ideal I is of the form $\mathcal{I}(V)$ for some affine variety V if and only if $I = \sqrt{I}$. Such ideals are called *radical ideals.* The operations $\mathcal{V}$ and $\mathcal{I}$ give mutually inverse bijections between the radical ideals of $\mathbf{K}[x_1, \dots, x_n]$ and affine varieties in $\mathbf{K}^n$.

The ring

$$\mathbf{K}[V] = \frac{\mathbf{K}[x_1, \dots, x_n]}{\mathcal{I}(V)}$$

is called the *coordinate ring* of V, and $\mathbf{K}[V]$ is a finitely generated $\mathbf{K}$-algebra with no nilpotent elements.

Let V and V' be two affine varieties. Let $\theta : V \to V'$ be any map from V to V'. Then any element $f \in \mathbf{K}[V']$ determines a map $\phi_f : V \to \mathbf{K}$ by putting $\phi_f(v) = f(\theta(v))$ for all $v \in V$. If $\phi_f \in \mathbf{K}[V]$ for all $f \in \mathbf{K}[V']$ then we have a map $\Phi : \mathbf{K}[V'] \to \mathbf{K}[V]$. If θ determines such a map Φ and if Φ is a $\mathbf{K}$-algebra homomorphism then we say that θ is a *morphism* of affine varieties. If moreover Φ is an isomorphism of $\mathbf{K}$-algebras, then we say θ is an *isomorphism* of affine varieties .

A set G is called an *affine algebraic group* over $\mathbf{K}$ if it satisfies the following:-

(i) G is an affine variety over $\mathbf{K}$;

(ii) G is a group;

(iii) the maps $G \times G \to G$ given by $(x, y) \mapsto xy$ and $G \to G$ given by $x \mapsto x^{-1}$ are morphisms of algebraic varieties.

Let G and G' be affine algebraic groups. A map $\alpha : G \to G'$ is a *homomorphism* of algebraic groups if α is both a morphism of affine varieties and a homomorphism of groups. If α is bijective and both α and α^{-1} are homomorphisms of algebraic groups then α is an *isomorphism* of algebraic groups.

If V is an affine variety then the *Zariski topology* on V is the topology in which the closed sets of V are the affine subvarieties of V. Hence any affine algebraic group G is a topological space, and is expressible as the disjoint union of its connected components. There are only finitely many such components and these are irreducible. Let G^0 be the component containing the identity element $1 \in G$. Then G^0 is a closed normal subgroup of G of finite index. The components of G are just the cosets xG^0 of G^0 in G. G^0 is a connected affine algebraic group, and is called the *connected component* of G.

An example of an affine algebraic group is $GL_n(\mathbf{K})$, the general linear group of degree n over $\mathbf{K}$. The closed subgroups of $GL_n(\mathbf{K})$ for various values of n are called the *linear algebraic groups,* and these are affine algebraic groups. In fact, every affine algebraic group is isomorphic to a closed subgroup of $GL_n(\mathbf{K})$ for some n. Thus the concepts of affine algebraic group and linear algebraic group coincide, and any affine algebraic group can be expressed in terms of matrices. We shall usually call such groups linear algebraic groups.

1.2 Semisimple and Reductive Groups.

Let G be a connected linear algebraic group over $\mathbf{K}$. Then the set of closed connected solvable normal subgroups of G has a unique maximal element. This is the product of all closed connected solvable normal subgroups of G, and is called the *radical* $R(G)$. Similarly the set of closed connected unipotent normal subgroups of G has a unique maximal element, called the *unipotent radical* $R_u(G)$. Every unipotent group is nilpotent. Thus $R_u(G) \subset R(G)$.

G is *semisimple* if $G \neq 1$ and $R(G) = 1$. G is *reductive* if $G \neq 1$ and $R_u(G) = 1$. Every semisimple group is reductive but the converse is not true. Suppose that G is a connected reductive group. Let $G' = [G, G]$ be its commutator subgroup and Z be its centre. Let Z^0 be the connected component of Z. Z^0 is called the *connected centre* of G. Then we have a factorisation of $G = G'Z^0$, where G' is a connected semisimple group and Z^0 is a torus. Both G' and Z^0 are normal subgroups of G, but $G' \cap Z^0$ is not necessarily equal to 1, and can be a non-trivial finite group. We say that G is an *almost direct product* of the semisimple group G' and the torus Z^0.

Due to this decomposition the study of reductive groups can largely be reduced to that of semisimple groups. Let G be a connected semisimple group. We can obtain an almost direct decomposition of G, this time into a product of simple groups. G is said to be *simple* if G has no proper closed connected normal subgroups. Any proper normal subgroup of a simple algebraic group must be finite and lie in the centre of the group. A connected semisimple group G has a finite set of closed normal subgroups $G_1, \ldots, G_k$ such that:-

(i) each G_i is simple;
(ii) $[G_i, G_j] = 1$ if $i \neq j$;
(iii) $G = G_1 \ldots G_k$;
(iv) $G_i \cap G_1 \ldots G_{i-1} G_{i+1} \ldots G_k$ is finite for each i.

The G_i are uniquely determined by these conditions, and are called the *simple components* of the semisimple group G. Thus each connected semisimple group is an almost direct product of simple groups.

The problem of understanding the structure of semisimple groups can in this way be largely reduced to that of simple groups.

1.3 Roots, Coroots and the Weyl Group.

Let G be a connected reductive linear algebraic group over $\mathbf{K}$. Let T be a maximal torus in G, and N its normalizer in G. Then the Weyl group of G is given by

$$W = \frac{N(T)}{T}$$

and is finite and independent of the choice of maximal torus T.

Let G_m denote the multiplicative group $\mathbf{K}^* = \mathbf{K} - \{0\}$ considered as a linear algebraic group. We define the *character group* $X(T)$ of T by

$$X(T) = \mathrm{Hom}(T, G_m)$$

as homomorphisms of algebraic groups. This is a commutative group under the operation

$$(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t)$$

for all $\chi_1, \chi_2 \in X(T)$ and all $t \in T$.

For some positive integer r

$$T \cong G_m \times \ldots \times G_m$$

with r factors. The only algebraic homomorphisms $G_m \to G_m$ are maps $\lambda \mapsto \lambda^n$ for all $\lambda \in \mathbf{K}$, and some $n \in \mathbf{Z}$. Thus $\mathrm{Hom}\,(G_m, G_m) \cong \mathbf{Z}$, and

$$X(T) = \mathrm{Hom}(G_m \times \ldots \times G_m, G_m) \cong \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$$

and so $X(T)$ is a free abelian group of rank r. We also call $X(T)$ the *weight lattice*, and refer to elements of $X(T)$ as *weights*.

Let $Y(T) = \mathrm{Hom}(G_m, T)$ be the set of algebraic homomorphisms of G_m into T. Then $Y(T)$ is a group under the operation

$$(\gamma_1 + \gamma_2)(\lambda) = \gamma_1(\lambda)\gamma_2(\lambda)$$

for all $\gamma_1, \gamma_2 \in Y$ and all $\lambda \in \mathbf{K}^*$. Then

$$Y(T) \cong \mathrm{Hom}(G_m, G_m \times \ldots \times G_m) \cong \mathbf{Z} \oplus \ldots \oplus \mathbf{Z},$$

and $Y(T)$ is a free abelian group of rank r and is called the group of cocharacters of T.

We define a bilinear map $(\ ,\ ) : X \times Y \to \mathbf{Z}$ as follows. If $\chi \in X(T)$ and $\gamma \in Y(T)$ then $\chi.\gamma \in \mathrm{Hom}(G_m, G_m)$. Hence for any $\lambda \in G_m$

$$(\chi.\gamma)(\lambda) = \lambda^n$$

for some $n \in \mathbf{Z}$. We define $(\chi, \gamma) = n$. This map is non-degenerate and gives rise to a duality between $X(T)$ and $Y(T)$, giving isomorphisms

$$X(T) \cong \mathrm{Hom}(Y(T), \mathbf{Z})$$
$$Y(T) \cong \mathrm{Hom}(X(T), \mathbf{Z}).$$

The Weyl group can be made to act on $X(T)$ and $Y(T)$ as follows. If $w \in W$ and $\chi \in X(T)$ we define ${}^w\chi \in X(T)$ by

$${}^w\chi(t) = \chi(t^w) \qquad t \in T.$$

Then $\chi \mapsto {}^w\chi$ is an automorphism of $X(T)$. If $\gamma \in Y(T)$ define $\gamma^w \in Y(T)$ by

$$\gamma^w(\lambda) = \gamma(\lambda)^w \qquad \lambda \in G_m.$$

Then $\gamma \mapsto \gamma^w$ is an automorphism of $Y(T)$. These two actions are related by the formula

$$(\chi, \gamma^w) = ({}^w\chi, \gamma)$$

for all $\chi \in X(T)$, $\gamma \in Y(T)$ and $w \in W$.

A maximal closed connected solvable subgroup of G is called a *Borel subgroup* and any two Borel subgroups are conjugate in G. Since T is closed connected and solvable then T must lie in some Borel subgroup B of G. Thus B has a semidirect product decomposition B = UT where $U = R_u(B)$, the unipotent radical of B. There is a unique Borel subgroup B^- containing T such that $B \cap B^- = T$. Then B and B^- are called *opposite* Borel subgroups. We have $B^- = U^-T$ where $U^- = R_u(B^-)$. Both U and U^- are connected groups normalized by T and satisfy $U \cap U^- = 1$. They are maximal unipotent subgroups of G.

Let G_a denote the additive group of $\mathbf{K}$ considered as an algebraic group. The only algebraic automorphisms of G_a are the maps $\lambda \mapsto \mu\lambda$ for some $\mu \in \mathbf{K}^*$. Thus $\mathrm{Aut}G_a \cong G_m$. We consider the minimal proper subgroups of U and U^- which are normalized by T. These are all connected unipotent groups of dimension 1, and so are isomorphic to G_a. T acts on each of them by conjugation, giving a homomorphism $T \to \mathrm{Aut}G_a$. Hence each of these 1-dimensional unipotent groups determines an element of $\mathrm{Hom}(T, G_m) = X(T)$. The elements of $X(T)$ arising in this way are called the *roots,* and are all non-zero. Distinct 1-dimensional unipotent subgroups give rise to distinct roots. The roots form a finite subset $\Phi \subset X(T)$, which is independent of the choice of Borel subgroup $B \supset T$. We denote by X_α the 1-dimensional subgroup which gives rise to the root $\alpha \in \Phi$ and call it a *root subgroup* of G. The roots arising from the root subgroups in U^- are the negatives of the roots arising from the root subgroups in U. We also have

$$G = < T, X_\alpha : \alpha \in \Phi > .$$

If G is a connected semisimple algebraic group then

$$G = < X_\alpha : \alpha \in \Phi > .$$

Let $\alpha, -\alpha \in \Phi$ be a pair of opposite roots, and consider the subgroup $< X_\alpha, X_{-\alpha} >$ of G generated by the root subgroups X_α and $X_{-\alpha}$. This subgroup is a 3-dimensional simple group isomorphic to either $SL_2(\mathbf{K})$ or to $PGL_2(\mathbf{K})$. There is a homomorphism $\phi : SL_2(\mathbf{K}) \to < X_\alpha, X_{-\alpha} >$ such that

$$\phi\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} = X_\alpha \qquad \phi\left\{\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\right\} = X_{-\alpha}.$$

Consequently, $\phi\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$ is a 1-dimensional subgroup of T. Let $\alpha^\nu : G_m \to T$ be the homomorphism given by

$$\alpha^\nu(\lambda) = \phi\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Then $\alpha^\nu \in \mathrm{Hom}(G_m, T) = Y(T)$ and is uniquely determined by $\alpha \in \Phi$. α^ν is called the *coroot* corresponding to the root α. They are related by the condition $< \alpha, \alpha^\nu > = 2$. The coroots from a finite subset of $Y(T)$ denoted by Φ^ν.

Let Φ^+ denote the set of roots arising from root subgroups of U and Φ^- denote those arising from root subgroups of U^-. Roots in Φ^+ and Φ^- are called *positive* and *negative* roots respectively. let $\Pi \subset \Phi$ be given by

$$\Pi = \{\alpha \in \Phi^+ : \alpha \neq \alpha' + \alpha'' \text{ for any } \alpha', \alpha'' \in \Phi^+\}.$$

Then Π is called the set of *simple roots* and is linearly independent. Let $l = |\Pi|$ and write $\Pi = \{\alpha_1, \dots, \alpha_l\}$. Each root $\alpha \in \Phi^+$ has the form

$$\alpha = \sum_{i=1}^{l} n_i \alpha_i \qquad n_i \in \mathbb{Z} \text{ and } n_i \geq 0,$$

and each root $\alpha' \in \Phi^-$ has the form

$$\alpha' = \sum_{i=1}^{l} n_i \alpha_i \qquad n_i \in \mathbb{Z} \text{ and } n_i \leq 0.$$

If $\alpha = \sum_{i=1}^{l} n_i \alpha_i \in \Phi$ then we define the *height* $\mathrm{ht}(\alpha)$ of α to be $\sum_{i=1}^{l} n_i$.

1.4 Rational Representations.

Let G be a linear algebraic group over $\mathbf{K}$. A *rational representation* of G is a homomorphism of algebraic groups from G into $GL_n(\mathbf{K})$ for some $n \in \mathbb{N}$. Since G is isomorphic to a closed subgroup of $GL_m(\mathbf{K})$ for some $m \in \mathbb{N}$, G can be expressed as a group of matrices. A representation

$$\rho : G \to GL_n(\mathbf{K})$$
$$\rho_{i,j} : g \mapsto \rho(g)_{i,j}$$

is rational if and only if $\rho_{i,j}(g)$ is a rational function in the coefficients of g, for all $i, j \in \{1, \dots, n\}$ and all $g \in G$.

The property of a representation being rational is independent of which expression of G in terms of matrices is chosen. Also any representation equivalent to a rational representation is also rational.

1.5 Irreducible Rational Representations and Dominant Weights.

Let $\mathbf{K}$ be an algebraically closed field of characteristic 0, and let G be a semisimple simply connected linear algebraic group over $\mathbf{K}$.

The group of characters $X(T)$ is a lattice over $\mathbb{Z}$, and is called the *weight lattice*. The weight lattice contains the root lattice $\mathbb{Z}\Phi$ as a subgroup of finite index. The set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_l\}$ forms a basis for the root lattice $\mathbb{Z}\Phi$, and so the corresponding set of coroots $\Pi^\nu = \{\alpha_1^\nu, \dots, \alpha_l^\nu\}$ forms a basis for the coroot lattice $\mathbb{Z}\Phi^\nu$, called the set of *fundamental coroots* . Since G is simply connected $\mathbf{Z}\Phi^{\nu} = Y(T)$. So there exist unique $\lambda_1, \ldots, \lambda_l \in X(T)$ such that

$$(\lambda_i, \alpha_j^{\nu}) = \delta_{i,j}$$

where $\delta_{i,j}$ is the Kronecker delta, for all $i, j \in \{1, \ldots, l\}$. The set $\{\lambda_1, \ldots, \lambda_l\}$ forms a basis for $X(T)$, and is called the set of *fundamental dominant weights.* A weight $\lambda_i \in X(T)$ is called *dominant* if

$$\lambda = \sum_{i=1}^{l} c_i \lambda_i \qquad c_i \in \mathbf{Z} \text{ and } c_i \geq 0.$$

This is equivalent to the condition

$$(\lambda, \alpha^{\nu}) \geq 0 \qquad \text{all } \alpha \in \Phi^+.$$

We denote the set of dominant weights by $X(T)^+$.

Let V be a G-module affording a rational representation $\rho : G \to GL(V)$. For $\lambda \in X(T)$ we define

$$V^{\lambda} = \{v \in V \; ; \; t.v = \lambda(t)v \text{ for all } t \in T\}.$$

Then V^{λ} is called the *weight space* of V corresponding to λ, and $\dim V^{\lambda}$ is the *multiplicity* of the weight λ. Those $\lambda \in X(T)$ such that $V^{\lambda} \neq 0$ are the weights of ρ, and we denote the set of them by $P(\rho)$ or $P(V)$. So

$$V = \bigoplus_{\lambda \in P(\rho)} V^{\lambda}.$$

Let V be an irreducible rational G-module affording the representation ρ. There is a unique 1-dimensional subspace of V stable under the action of B, and it consists of all vectors in V fixed under the action of U. This subspace forms a T-module and the corresponding weight λ_ρ is dominant. All other weights of ρ are of the form

$$\lambda_\rho - \sum_{\alpha \in \Pi} c_\alpha \alpha \quad c_\alpha \in \mathbb{N}.$$

Any two irreducible rational representations ρ and ρ' of G are equivalent if and only if $\lambda_\rho = \lambda_{\rho'}$.

There is a partial ordering on the weight lattice $X(T)$ given by $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots. So we see that λ_ρ is strictly greater than all other weights in $P(\rho)$, and thus λ_ρ is called the *highest weight* of V. An element $v_{\lambda_\rho} \in V^{\lambda_\rho} - \{0\}$ is called a *highest weight vector* .

1.6 The Borel-Weil Theorem and Dual Modules.

Let $\mathbf{K}$ be an algebraically closed field of characteristic 0, and let G be a semisimple simply connected linear algebraic group over $\mathbf{K}$. The Borel-Weil theorem states that for every dominant weight $\lambda \in X(T)^+$ there is an irreducible rational G-module of highest weight λ. This is constructed as follows. Define a subspace $F_\lambda \subset \mathbf{K}[G]$ by

$$F_\lambda = \{f \in \mathbf{K}[G]; f(bg) = \lambda(b)f(g) \text{ for all } b \in B^-, g \in G\}.$$

This is a G-module under the action

$$(g.f)(x) = f(xg) \quad f \in F_\lambda \text{ and } g, x \in G,$$

and is an irreducible rational module of highest weight λ.

From this and the preceding section we see that there is a one-to-one correspondence between the set of dominant weights of G and the set of irreducible rational G-modules.

Let V be an irreducible rational G-module of highest weight λ. The dual space V^* also affords a rational representation of G, where G acts according to the rule

$$(g.f)(v) = f(g^{-1}v) \quad g \in G, f \in V^*, v \in V.$$

V^* must also be irreducible and we are interested in its highest weight. The Weyl group acts on G by conjugation and there is a unique element $\sigma_0 \in W$ which interchanges B and B^-. The highest weight of V^* is $-\sigma_0(\lambda)$.

It is known that $\sigma_0 = -1$ for the irreducible root systems A_1, B_l, C_l, D_l for l even, E_7, E_8, F_4 and G_2. In these cases V^* is always isomorphic (as a G-module) to V.

1.7 Weyl's Dimension Formula.

In this section we describe a formula, due to Weyl [1], for the dimension of an irreducible rational module of a semisimple simply connected algebraic group G over an algebraically closed field $\mathbf{K}$ of characteristic 0.

Suppose F_λ is an irreducible rational G-module of highest weight $\lambda \in X(T)^+$. Let Φ^+ denote the set of positive roots of G as before. Let ρ be given by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi +} \alpha.$$

Then $\rho \in X(T)$ since $\rho = \lambda_1 + \ldots \lambda_l$. The dimension of F_λ is given by

$$\dim F_\lambda = \prod_{\alpha \in \Phi +} \frac{(\lambda + \rho, \alpha^\nu)}{(\rho, \alpha^\nu)}$$

where $(\ ,\ ) : X(T) \times Y(T) \to \mathbf{Z}$ is the bilinear map defined in Section 1.3.

1.8 The Lie Algebra.

Let $\mathbf{K}$ be an algebraically closed field of characteristic 0, and let G be a linear algebraic group over $\mathbf{K}$. The group variety is non-singular and so at every point a tangent space is defined. Let $\mathfrak{g}$ be the tangent space at the identity element of G. Since the variety has a group structure the tangent space has the structure of a Lie algebra, and we write $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G.

If G is a simple algebraic group then $\mathrm{Lie}(G)$ is a simple Lie algebra, and if G is semisimple then $\mathrm{Lie}(G)$ is semisimple. We have mentioned various subgroups of G and there are corresponding Lie algebras to these. The Lie algebra of the torus $\mathrm{Lie}(T) = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. For each root subgroup X_α there is a corresponding 1-dimensional subspace $\mathbf{K}e_\alpha = \mathrm{Lie}(X_\alpha)$. The subspaces $\mathfrak{n}^+ = \sum_{\alpha\in\Phi^+} \mathbf{K}e_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha\in\Phi^-} \mathbf{K}e_\alpha$ are maximal nilpotent subalgebras of $\mathfrak{g}$ satisfying $\mathrm{Lie}(U) = \mathfrak{n}^+$ and $\mathrm{Lie}(U^-) = \mathfrak{n}^-$. Furthermore $\mathfrak{b}^+ = \mathfrak{h}+\mathfrak{n}^+ = \mathrm{Lie}(B)$ and $\mathfrak{b}^- = \mathfrak{h}+\mathfrak{n}^- = \mathrm{Lie}(B^-)$ are maximal solvable subalgebras of $\mathfrak{g}$. In this way we obtain the Cartan decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Phi} \mathbf{K}e_\alpha.$$

1.9 The Universal Enveloping Algebra.

Given any Lie algebra $\mathfrak{g}$ over $\mathbf{K}$ there is an associative $\mathbf{K}$-algebra $\mathcal{U}(\mathfrak{g})$ in which $\mathfrak{g}$ can be embedded. $\mathcal{U}(\mathfrak{g})$ is called the *universal enveloping algebra* of $\mathfrak{g}$ and is defined to be the quotient of the tensor algebra $T(\mathfrak{g}) = \sum_{i\geq 0} T^i(\mathfrak{g})$ by the ideal J generated by the elements $(x\otimes y - y\otimes x - [x,y])$ for all $x, y \in \mathfrak{g}$.

The following is a basic result about $\mathcal{U}(\mathfrak{g})$.

Poincaré-Birkhoff-Witt basis Theorem.

The canonical map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective, and $\mathfrak{g}$ may be identified with $i(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$. Let $(x_1, \ldots, x_t)$ be any ordered basis of $\mathfrak{g}$. Then the elements

$$x_1^{\alpha_1} \ldots x_t^{\alpha_t}$$

for $\alpha_1, \ldots, \alpha_t \in \mathbb{Z}$ such that $\alpha_1, \ldots, \alpha_t \geq 0$ form a basis for $\mathcal{U}(\mathfrak{g})$. We will refer to a basis of this type as a PBW basis.

Representations.

Suppose that we have a homomorphism of algebraic groups $\phi : G \to GL_m(\mathbf{K})$ for some $m \in \mathbb{N}$. The differential $d\phi : \mathfrak{g} \to \mathfrak{gl}_m(\mathbf{K})$ where $\mathfrak{gl}_m(\mathbf{K}) = \mathrm{Lie}(GL_m(\mathbf{K}))$, is a homomorphism of Lie algebras.

In this way any representation $\rho : G \to GL_m(V)$ of G into the group of non-singular linear maps of a vector space V over $\mathbf{K}$ determines a representation $\rho' = d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ into the Lie algebra of endomorphisms of V. Moreover, ρ is irreducible if and only if $d\rho$ is irreducible. So G and $\mathfrak{g}$ have the same set of irreducible modules.

The representation $\rho' : \mathfrak{g} \to \mathfrak{gl}(V)$ extends canonically to a homomorphism of $\mathcal{U}(\mathfrak{g})$ into the associative algebra of endomorphisms of V and makes V into a $\mathcal{U}(\mathfrak{g})$-module. We will call the representation of $\mathcal{U}(\mathfrak{g})$ obtained in this way ρ' as well.

1.10 $\mathbf{Z}$-Forms.

We briefly introduce the subject of constructing $\mathfrak{g}$ and its representation over $\mathbf{Z}$. Assume that $\mathfrak{g}$ is semisimple.

For $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$, let $p(\alpha,\beta) \in \mathbf{Z}$ be the greatest integer such that $\alpha - p(\alpha,\beta)\beta \in \Phi$. According to Chevalley [1] it is possible to choose $e_\alpha \in \mathrm{Lie}(X_\alpha) - \{0\}$ such that if $h_\alpha = [e_\alpha, e_{-\alpha}]$ then :-

(i) $[e_\alpha, e_\beta] = 0$ for all $\alpha, \beta \in \Phi$, such that $\alpha + \beta \notin \Phi$ and $\alpha + \beta \neq 0$;

(ii) $[e_\alpha, e_\beta] = \pm(p(\alpha,\beta)+1)e_{\alpha+\beta}$ for all $\alpha, \beta \in \Phi$, such that $\alpha + \beta \in \Phi$;

(iii) $[h_\alpha, e_\beta] = (\beta, \alpha^\nu)e_\beta$ for all $\alpha, \beta \in \Phi$.

The elements $h_\alpha, \alpha \in \Pi$ and $e_\beta, \beta \in \Phi$ form a basis of $\mathfrak{g}$, called a *Chevalley basis*. The $\mathbf{Z}$-span $\mathfrak{g}_\mathbf{Z}$ of this basis is a lattice in $\mathfrak{g}$ independent of the choice of Π. It is also a Lie algebra over $\mathbf{Z}$ under the bracket operation inherited from $\mathfrak{g}$. We say that $\mathfrak{g}_\mathbf{Z}$ is a $\mathbf{Z}$-form of $\mathfrak{g}$ because $\mathfrak{g} = \mathbf{K} \otimes_\mathbf{Z} \mathfrak{g}_\mathbf{Z}$. Although $\mathfrak{g}_\mathbf{Z}$ depends on the choice of the e_α it is determined up to isomorphism by $\mathfrak{g}$ alone. For any field F define $\mathfrak{g}_F = F \otimes_\mathbf{Z} \mathfrak{g}_\mathbf{Z}$.

The Lie algebra acts on itself by left (bracket) multiplication, and for any $a \in \mathfrak{g}$ the map

$$(\mathrm{ad}\ a) : x \mapsto [a, x] \quad x \in \mathfrak{g}$$

is called the *adjoint mapping* determined by a. For any $\alpha \in \Phi$ and any $n \in \mathbb{N}$ the map $\frac{1}{n!}(\mathrm{ad}\ a)^n$ leaves the lattice $\mathfrak{g}_\mathbf{Z}$ invariant.

Assume G is semisimple and simply-connected. For $\alpha \in \Phi$ and $t \in \mathbf{K}$ define a map $x_\alpha(t) : \mathfrak{g} \to \mathfrak{g}$ by

$$\begin{aligned} x_\alpha(t) &= 1 + t(\mathrm{ad}\ e_\alpha) + \frac{t^2(\mathrm{ad}\ e_\alpha)^2}{2!} + \frac{t^3(\mathrm{ad}\ e_\alpha)^3}{3!} + \dots \\ &= \exp(t(\mathrm{ad}\ e_\alpha)) \end{aligned}$$

since $(ad\ e_\alpha)$ is nilpotent. Then we define the *adjoint group* G_{ad} of the same type as G by

$$G_{ad} = < x_\alpha(t); \alpha \in \Phi, t \in \mathbf{K} > .$$

We will now construct the *Kostant* $\mathbf{Z}$*-form* $\mathcal{U}_\mathbf{Z} \subset \mathcal{U}(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$. Let $m = |\Phi^+|$ and write $\Phi^+ = \{\beta_1, \dots, \beta_m\}$. As before let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ and denote h_{α_i} by h_i. Let $\mathbf{Z}_+$ denote the set of non-negative integers. For $Q = (q_1, \dots, q_m) \in (\mathbf{Z}_+)^m$

we define the elements $e_Q, f_Q \in \mathcal{U}(\mathfrak{g})$ by

$$e_Q = \prod_{i=1}^{m} \frac{(e_{\beta_i})^{q_i}}{q_i!} \quad \text{and}$$

$$f_Q = \prod_{i=1}^{m} \frac{(e_{-\beta_i})^{q_i}}{q_i!}.$$

Given $x \in \mathfrak{g}$ and $s \in \mathbb{N}$ define $\binom{x}{s} \in \mathcal{U}(\mathfrak{g})$ by

$$\binom{x}{s} = \frac{x(x-1)\dots(x-s+1)}{s!}.$$

Then for $R = (r_1, \dots, r_l) \in (\mathbb{Z}_+)^l$ define the element $h_R \in \mathcal{U}(\mathfrak{h})$ by

$$h_R = \prod_{i=1}^{l} \binom{h_i}{r_i}.$$

Kostant proves in [1] that the elements

$$f_Q h_R e_S$$

for all $Q, S \in (\mathbb{Z}_+)^m$ and $R \in (\mathbb{Z}_+)^l$, form a basis of a $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}} \subset \mathcal{U}(\mathfrak{g})$ analogous to a PBW basis for $\mathcal{U}(\mathfrak{g})$. Also the elements f_Q (respectively h_R or e_S) form the basis of a $\mathbb{Z}$-form $\mathcal{U}(\mathfrak{n}^-)_{\mathbb{Z}} \subset \mathcal{U}(\mathfrak{n}^-)$ (respectively $\mathcal{U}(\mathfrak{h})_{\mathbb{Z}} \subset \mathcal{U}(\mathfrak{h})$ or $\mathcal{U}(\mathfrak{n}^+)_{\mathbb{Z}} \subset \mathcal{U}(\mathfrak{n}^+)$). The lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ is stable under the action of $\mathcal{U}_{\mathbb{Z}}$.

Admissible Lattices.

Let V be a finite dimensional rational G-module. An *admissible* $\mathbb{Z}$-form of V is a $\mathbb{Z}$-form which is stable under the action of $\mathcal{U}_{\mathbb{Z}}$. An admissible $\mathbb{Z}$-form of V always exists, and moreover, if $V_{\mathbb{Z}} \subset V$ is an admissible $\mathbb{Z}$-form then

$$V_{\mathbb{Z}} = \bigoplus_{\mu \in P(V)} V_{\mathbb{Z}}^{\mu}$$

where $V_{\mathbb{Z}}^{\mu} = V_{\mathbb{Z}} \cap V^{\mu}$ for all $\mu \in P(V)$, the set of weights in V.

If V is an irreducible rational G-module of highest weight λ, and v_λ is a highest weight vector, then we can obtain an admissible $\mathbb{Z}$-form $V_{\mathbb{Z}} \subset V$ by putting

$$V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} v_\lambda = \mathcal{U}(\mathfrak{n}^-)_{\mathbb{Z}} v_\lambda.$$

1.11 Chevalley Groups Over Arbitrary Fields.

Recall the map $x_\alpha(t) : \mathfrak{g} \to \mathfrak{g}$ defined for any $\alpha \in \Phi$ and any $t \in \mathbf{K}$ by

$$x_\alpha(t) = \exp(\mathrm{ad}\ te_\alpha).$$

Let $\mathbf{k}$ be an arbitrary field, and let $t \in \mathbf{k}$. Then $\exp(\mathrm{ad}\ te_\alpha)$ is a map from $\mathfrak{g}_\mathbf{k}$ to itself, where $\mathfrak{g}_\mathbf{k} = \mathbf{k} \otimes_\mathbf{Z} \mathfrak{g}_\mathbf{Z}$ as before. The matrix group generated by maps $\exp(\mathrm{ad}\ te_\alpha)$, for all $\alpha \in \Phi$ and $t \in \mathbf{k}$ consists of non-singular matrices with coefficients in $\mathbf{k}$. Such a group is called the *(adjoint) Chevalley group* over $\mathbf{k}$ of the same type as G, and is denoted by $G_\mathbf{k}$. Define $G_\mathbf{Z}$ to be the matrix group over $\mathbf{Z}$ generated by maps $\exp(n(\mathrm{ad}\ e_\alpha))$ for all $n \in \mathbf{Z}$ and $\alpha \in \Phi$. Then the $\mathbf{Z}$-form $\mathfrak{g}_\mathbf{Z} \subset \mathfrak{g}$ is stable under the action of $G_\mathbf{Z}$.

There is a more general construction of Chevalley groups which we briefly describe here. Let V be a faithful G-module, and denote the $\mathbf{Z}$-lattice of weights in V by $\Lambda(V)$. Then $\mathbf{Z}\Phi \subset \Lambda(V) \subset X(T)$. Let $V_\mathbf{Z}$ be an admissible lattice in V. Let $\mathbf{k}$ be an arbitrary field, and let $V_\mathbf{k} = \mathbf{k} \otimes_\mathbf{Z} V_\mathbf{Z}$.

Since V is a G-module, it is also a $\mathfrak{g}$-module, and so $V_\mathbf{k}$ is a $\mathfrak{g}_\mathbf{k}$-module, where $\mathfrak{g}_\mathbf{k}$ is defined by $\mathfrak{g}_\mathbf{k} = \mathbf{k} \otimes_\mathbf{Z} \mathfrak{g}_\mathbf{Z}$ as before. Hence $V_\mathbf{k}$ is stable under the action of $1 \otimes \left(\frac{(\mathrm{ad} e_\alpha)^n}{n!}\right)$, for all $n \in \mathbb{N}$, which we denote by $e_{\alpha,n}$. We also define $e_{\alpha,0}$ to be the identity. We have $e_{\alpha,n} = 0$ for large enough n, so for any $t \in \mathbf{k}$ we can define $\theta_\alpha(t) \in \mathrm{End}_\mathbf{k}(V_\mathbf{k})$ by $\theta_\alpha(t) = \sum_{n=0}^{\infty} t^n e_{\alpha,n}$. The group generated by all $\theta_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathbf{k}$, is denoted by $G_V(\mathbf{k})$. Although $G_V(\mathbf{k})$ depends on the choice of admissible lattice $V_\mathbf{Z}$, it only depends up to isomorphism on the weight lattice $\Lambda(V)$ of V.

If $V = \mathfrak{g}$ then $\Lambda(V) = \mathbf{Z}\Phi$, and $G_\mathfrak{g}(\mathbf{k})$ constructed using the admissible lattice $\mathfrak{g}_\mathbf{Z}$ is the same as the adjoint Chevalley group $G_\mathbf{k}$ defined above. If the module V satisfies $\Lambda(V) = X(T)$ then $G_V(\mathbf{k})$ is called the *universal Chevalley group.*

1.12 Irreducible Rational Representations in Characteristic p.

Consider the case when the field $\mathbf{K}$ is algebraically closed and has characteristic $p > 0$. Let G be a connected semisimple simply connected linear algebraic group over $\mathbf{K}$.

The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ decomposes in the same way as in the characteristic zero case, namely

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Phi} \mathbf{K} e_\alpha$$

where $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ such that there is a maximal torus $T \subset G$ with $\mathrm{Lie}(T) = \mathfrak{h}$. The lattice of weights $X(T)$ can be defined in the same way, along with the set of dominant weights $X(T)^+ \subset X(T)$.

As in the characteristic zero case every irreducible rational representation of G has a unique highest weight which is dominant, and any two irreducible G-modules of the same highest weight are isomorphic.

So we still have a one-to-one correspondence between the set of dominant weights and the set of irreducible rational G-modules. However, the dimensions of these modules are no longer given by Weyl's dimension formula. In general their dimensions are not known.

1.13 Weyl Modules.

Let $G_{\mathbf{C}}$ be a semisimple simply-connected linear algebraic group over $\mathbf{C}$. Let $\mathbf{k}$ be an infinite field of characteristic $p > 0$, and let $\overline{G_{\mathbf{k}}}$ denote the universal Chevalley group of G over $\mathbf{k}$. Let $\mathfrak{g}_{\mathbf{C}} = \mathrm{Lie}(G_{\mathbf{C}})$ and let $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ be its enveloping algebra. If $\mathcal{U}_{\mathbf{Z}}$ is the Kostant $\mathbf{Z}$-form of $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ then define the *hyperalgebra* of $\overline{G_{\mathbf{k}}}$ by

$$\mathcal{U}_{\mathbf{k}} = \mathbf{k} \otimes_{\mathbf{Z}} \mathcal{U}_{\mathbf{Z}}.$$

The hyperalgebra $\mathcal{U}_{\mathbf{k}}$ plays the role of the enveloping algebra in the characteristic p situation. Recall that $\mathcal{U}_{\mathbf{Z}} = \mathcal{U}(\mathfrak{n}^-)_{\mathbf{Z}}\mathcal{U}(\mathfrak{h})_{\mathbf{Z}}\mathcal{U}(\mathfrak{n}^+)_{\mathbf{Z}}$ and so

$$\mathcal{U}_{\mathbf{K}} = \mathcal{U}(\mathfrak{n}^-)_{\mathbf{k}}\mathcal{U}(\mathfrak{h})_{\mathbf{k}}\mathcal{U}(\mathfrak{n}^+)_{\mathbf{K}}$$

where $\mathcal{U}(\mathfrak{n}^-)_{\mathbf{k}} = \mathbf{k} \otimes_{\mathbf{Z}} \mathcal{U}(\mathfrak{n}^-)_{\mathbf{Z}}$, $\mathcal{U}(\mathfrak{h})_{\mathbf{k}} = \mathbf{k} \otimes_{\mathbf{Z}} \mathcal{U}(\mathfrak{h})_{\mathbf{Z}}$ and $\mathcal{U}(\mathfrak{n}^+)_{\mathbf{k}} = \mathbf{k} \otimes_{\mathbf{Z}} \mathcal{U}(\mathfrak{n}^+)_{\mathbf{Z}}$ are three subalgebras of $\mathcal{U}_{\mathbf{k}}$. There is a one-to-one correspondence between finite-dimensional rational $\overline{G_{\mathbf{k}}}$-modules and finite-dimensional $\mathcal{U}_{\mathbf{k}}$-modules (see Sullivan [1] and [2]).

For each $\lambda \in X(T)^+$ we wish to construct a rational module for $\overline{G_{\mathbf{k}}}$ with dimension given by Weyl's dimension formula with respect to λ. An irreducible rational $G_{\mathbf{C}}$-module $V_{\lambda,\mathbf{C}}$ is uniquely determined up to isomorphism by $\lambda \in X(T)^+$ and has the required dimension.

Let $v_\lambda \in V_{\lambda,\mathbf{C}}$ be a highest weight vector. Then $V_{\lambda,\mathbf{Z}} = \mathcal{U}_{\mathbf{Z}} v_\lambda = \mathcal{U}(\mathfrak{n}^-)_{\mathbf{Z}} v_\lambda$ is an admissible lattice in $V_{\lambda,\mathbf{C}}$. We obtain a $\overline{G_{\mathbf{k}}}$-module $V_{\lambda,\mathbf{k}}$ by taking the tensor product

$$V_{\lambda,\mathbf{k}} = \mathbf{k} \otimes_{\mathbf{Z}} V_{\lambda,\mathbf{Z}}.$$

Let $\overline{v_\lambda} = 1 \otimes v_\lambda \in V_{\lambda,\mathbf{k}}$. Then, $V_{\lambda,\mathbf{k}} = \mathcal{U}_{\mathbf{k}}\overline{v_\lambda} = \mathcal{U}(\mathfrak{n}^-)_{\mathbf{k}}\overline{v_\lambda}$. The module $V_{\lambda,\mathbf{k}}$ is rational and is called a *Weyl module,* because it has dimension given by Weyl's dimension formula with respect to λ. $V_{\lambda,\mathbf{C}}$ is irreducible, but when the characteristic of $\mathbf{k}$ is $p > 0$ then $V_{\lambda,\mathbf{k}}$ is not in general irreducible. However the sum

$$\sum_{\substack{\mu \in P(V) \\ \mu \neq \lambda}} (V_{\lambda,\mathbf{k}})^\mu$$

of all the weight spaces in $V_{\lambda,\mathbf{k}}$ of weight $\mu \neq \lambda$ is a proper $\mathbf{k}$-subspace of $V_{\lambda,\mathbf{k}}$ and since $\overline{v_\lambda}$ generates $V_{\lambda,\mathbf{k}}$ all proper submodules must lie inside it. Hence the sum of all proper submodules of $V_{\lambda,\mathbf{k}}$ is still a proper submodule, which is denoted by $M_{\lambda,\mathbf{k}}$, and does not contain $\overline{v_\lambda}$. Let $F_{\lambda,\mathbf{k}}$ be the factor module $V_{\lambda,\mathbf{k}}/M_{\lambda,\mathbf{k}}$. Then $F_{\lambda,\mathbf{k}}$ is an irreducible rational $G_{\mathbf{k}}$-module of highest weight λ and as λ ranges over all dominant weights the set of $F_{\lambda,\mathbf{k}}$ forms a complete set of irreducible rational $G_{\mathbf{k}}$-modules.

2

General Linear Groups.

Let $\mathbf{k}$ be an infinite field and let $n \in \mathbb{N}$. Let $G = GL_n(\mathbf{k})$ the group of non-singular $n \times n$ matrices over $\mathbf{k}$. In this chapter we summarise the work of Green [1] on polynomial representations of G.

2.1 Polynomial and Rational Representations of $GL_n(\mathbf{k})$.

A rational representation $\rho : G \to GL_m(\mathbf{k})$ for some $m \in \mathbb{N}$ is a group homomorphism such that each coordinate function $\rho_{i,j} : g \mapsto \rho(g)_{i,j}$ is a rational function in the coefficients of g, independent of $g \in G$.

Let $i, j \in \{1, \ldots, n\}$. Define the *coefficient function*

$$\begin{aligned} c_{i,j} : G &\to \mathbf{k} \\ g &\mapsto g_{i,j} \end{aligned}$$

to be the map which sends a matrix to its $(i,j)^{\text{th}}$-coefficient. Then any rational representation $\rho : G \to GL_m(\mathbf{k})$, for some $m \in \mathbb{N}$, satisfies, for each $i, j \in \{1, \ldots, n\}$,

$$\rho_{i,j} = \frac{f_{i,j}}{g_{i,j}}$$

where $f_{i,j}$ and $g_{i,j}$ are polynomials in the coefficient functions, and $g_{i,j}(g) \neq 0$ whenever $\det(g) \neq 0$. In fact, $g_{i,j}$ must be equal to a power of the determinant function.

A *polynomial representation* of G is a group homomorphism $\rho : G \to GL_m(\mathbf{k})$ for some $m \in \mathbb{N}$, such that each coordinate function $\rho_{i,j}$ is a polynomial in the coefficient functions. A *polynomial G-module* is a G-module which affords a polynomial representation. A submodule or quotient module of a polynomial G-module is again a polynomial G-module.

If all the coordinate functions $\rho_{i,j}$ of a polynomial representation ρ are homogeneous of the same degree, ρ is called *homogeneous,* and a G-module affording a homogeneous polynomial representation is called *homogeneous.* Any polynomial representation of G is equivalent to a direct sum of homogeneous representations. So, in studying polynomial G-modules, it is enough to consider only homogeneous ones. Let $r \in \mathbb{N}$. We will describe two ways of constructing homogeneous G-modules of degree r.

2.2 Partition Diagrams and Tableaux.

The subset of diagonal matrices $S \subset G$ forms a maximal torus in G isomorphic to a direct sum of n copies of the multiplicative group of the field. A weight $\lambda \in X(S)$ of S is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n integers describing the map

$$\lambda : \begin{pmatrix} s_1 & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{pmatrix} \mapsto s_1^{\lambda_1} s_2^{\lambda_2} \ldots s_n^{\lambda_n}$$

where $s_1, s_2, \ldots, s_n \in \mathbf{k} \setminus \{0\}$. Let $\lambda, \mu \in X(S)$. The sum $\lambda + \mu \in X(S)$ is given by

$$(\lambda + \mu)(s) = \lambda(s)\mu(s) \qquad \text{for all } s \in S,$$

and so $\lambda + \mu = (\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n)$. The set of dominant weights $X(S)^+$ for G is the set of integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of at most n parts, satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

For the remainder of this chapter fix an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ and $\lambda_1 + \ldots + \lambda_n = r$.

2.2.1 Definitions. A λ-*diagram* consists of r squares arranged in consecutive rows so that the first row has λ_1 squares, the second row has λ_2 squares and so on. The rows are counted from top to bottom and are aligned on the left. The columns are counted from left to right, and the i^{th} column has μ_i squares, where μ is the conjugate partition to λ.

A λ-tableau is a distribution of r numbers (or symbols), which are not necessarily distinct, in the r squares of a λ-diagram, one entry in each square.

The λ-tableau which contains all the numbers from 1 to r entered in order down the columns starting on the left and moving down consecutive columns from left to right is called the *leading λ-tableau* (or *leading standard λ-tableau* in Carter and Lusztig [1]).

For example, the leading $(3,2)$-tableau is

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2} \end{array}\,.$$

A λ-tableau is said to be *semistandard* if its numbers increase strictly from top to bottom in each column and do not decrease from left to right along each row (these are called *standard tableaux* in Green [1]). The semistandard λ-tableau which for all i has the number i in each square of the i^{th} row is called the *basic λ-tableau* (Carter and Lusztig [1] call this the *leading semistandard λ-tableau*). When there is no confusion as to λ this tableau is denoted by T_0. The basic $(2,2,1)$-tableau is

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}\,.$$

Define the i^{th} *position* in a λ-tableau to be the square in which the leading λ-tableau has the number i. Let T be a λ-tableau and let $\sigma \in S_r$, where S_r is the group of permutations on r elements. Let t_i denote the number in the i^{th} position of T. Define the λ-tableau $\sigma(T)$ to be the tableau in which the i^{th} position contains the number $t_{\sigma^{-1}(i)}$. In this way, the symmetric group S_r acts on the set of λ-tableaux with entries from $\{1,\ldots,n\}$.

2.3 Weyl Modules.

We describe the first of two ways to construct a homogeneous polynomial G-module of degree r and of highest weight λ. These results can be found in Carter and Lusztig [1].

Let V be a vector space over $\mathbf{k}$ of dimension n. G acts on the left of V in the normal way. Let $T^r(V)$ denote the r^{th} tensor power of V with itself, that is

$$T^r(V) = V \otimes_{\mathbf{k}} V \otimes_{\mathbf{k}} \ldots \otimes_{\mathbf{k}} V \qquad (r \text{ times}).$$

$T^r(V)$ is a left G-module under the action of G on each component separately. Denote by $T(V)$ the tensor algebra $T(V) = \sum_{s\geq 0} T^s(V)$. Let S_r be the group of permutations on r elements. Then S_r acts on the left of $T^r(V)$ via

$$\sigma.(v_1 \otimes v_2 \otimes \ldots \otimes v_r) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(r)}$$

where $\sigma \in S_r$ and $v_1,\ldots,v_r \in V$.

Let μ be the conjugate partition to λ, and so μ has λ_1 parts. For $h \in \{1,\ldots,\lambda_1\}$ define the subset $I_h \subset \{1,\ldots,r\}$ by

$$I_h = \{i \in \mathbb{N} : \mu_1 + \ldots + \mu_{h-1} < i \leq \mu_1 + \ldots + \mu_h\}.$$

In this way μ defines a decomposition of the set $\{1,\ldots,r\}$ into disjoint subsets I_1, I_2, $\ldots$, I_{λ_1}, which are the columns of the leading λ-tableau. Let $V^* = Hom(V,\mathbf{k})$. There is a canonical pairing $T^r(V) \times T^r(V^*) \to \mathbf{k}$ which will be denoted by $< \, , \, >$.

2.3.1 Definition. Let $V_{\lambda,\mathbf{k}} \subset T^r(V)$ be the subset of vectors $t \in T^r(V)$ which satisfy the following:-

(i)

$$< t \, , \, v_1' \otimes v_2' \otimes \ldots \otimes v_r' >= 0$$

whenever $v_1', v_2', \ldots, v_r' \in V^*$ are such that there exist $i \neq j$ in the same subset I_h with $h \in \{1,\ldots,\lambda_1\}$ such that $v_i' = v_j'$;

(ii) $(1 + (i\ j))t = 0$ for all $i,j \in I_h$ with $i \neq j$ and all $h \in \{1,\ldots,\lambda_1\}$;

(iii) for all $h \in \{1,\ldots,\lambda_1 - 1\}$ and all subsets $J_h \subset I_h$ and $J_{h+1} \subset I_{h+1}$ such that $|J_h| + |J_{h+1}| > |I_h|$ we have

$$\sum_{\sigma \in S(J)} \operatorname{sign}(\sigma)\sigma(t) = 0$$

where σ runs over the set $S(J)$ of all permutations of $\{1, 2, \ldots, r\}$ which are the identity outside $J_h \cup J_{h+1}$ and such that $\sigma(i) < \sigma(j)$ for $i < j$ in J_h and for $i < j$ in J_{h+1}.

Notice that condition (ii) is vacuously satisfied when $I_{h+1} = \emptyset$. Equations of the form described in condition (ii) are called *Garnir column relations.*

$V_{\lambda,\mathbf{k}}$ is a polynomial G-module because it is a submodule of $T^r(V)$, and it is called a *Weyl module.*

2.3.2 Definitions. Let $s \in \mathbb{N}$ and let $v_1, \ldots, v_s \in V$. These determine an element of $T^s(V)$, similar to a determinant, given by

$$\begin{vmatrix} v_1 & v_1 & \ldots & v_1 \\ v_2 & v_2 & \ldots & v_2 \\ \vdots & \vdots & & \vdots \\ v_s & v_s & \cdots & v_s \end{vmatrix} = \sum_{\sigma \in S_s} \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(s)}.$$

This is not zero, as it might at first appear, since tensor products are not commutative. As an example let $s = 3$ and $v_1, v_2, v_3 \in V$. Writing $v_1 v_2 v_3$ instead of $v_1 \otimes v_2 \otimes v_3$ we have

$$\begin{vmatrix} v_1 & v_1 & v_1 \\ v_2 & v_2 & v_2 \\ v_3 & v_3 & v_3 \end{vmatrix} = v_1 v_2 v_3 - v_2 v_1 v_3 - v_3 v_2 v_1 - v_1 v_3 v_2 + v_3 v_1 v_2 + v_2 v_3 v_1.$$

Denote the subgroup of permutations which leave invariant the columns of a λ-tableau by $C(\lambda) \subset S_r$. For example, if $\lambda = (2, 2)$ then $r = 4$ and

$$C(\lambda) = \{\ 1\ ,\ (1\ 2)\ ,\ (3\ 4)\ ,\ (1\ 2)(3\ 4)\ \}.$$

Let $\alpha \in \mathbb{Z}S_r$ be the element in the group algebra of S_r over $\mathbb{Z}$ given by

$$\alpha = \sum_{\sigma \in C(\lambda)} \operatorname{sign}(\sigma)\sigma.$$

In the above example

$$\alpha = 1 - (1\ 2) - (3\ 4) + (1\ 2)(3\ 4).$$

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and let $V_{\mathbb{Z}}$ be the free $\mathbb{Z}$-module of rank n generated by $\{v_1, v_2, \ldots, v_n\}$. Let T be a λ-tableau with entries from $\{1, \ldots, n\}$. If we denote by t_i the number in the i^{th} position of T then T determines a tensor $t_T \in T^r(V_{\mathbb{Z}})$ by

$$t_T = v_{t_1} \otimes v_{t_2} \otimes \ldots \otimes v_{t_r}.$$

Let T_0 be the basic λ-tableau as before. We define $\phi_\lambda \in T^r(V_{\mathbb{Z}})$ to be the tensor given by

$$\phi_\lambda = \alpha t_{T_0} = \begin{vmatrix} v_1 & v_1 & v_1 & \ldots & v_1 \\ v_2 & v_2 & v_2 & \ldots & v_2 \\ v_3 & v_3 & v_3 & \ldots & v_3 \\ \vdots & \vdots & \vdots & & \vdots \\ v_{\mu_1} & v_{\mu_1} & v_{\mu_1} & \cdots & v_{\mu_1} \end{vmatrix} \otimes \begin{vmatrix} v_1 & v_1 & \ldots & v_1 \\ v_2 & v_2 & \ldots & v_2 \\ \vdots & \vdots & & \vdots \\ v_{\mu_2} & v_{\mu_2} & \cdots & v_{\mu_2} \end{vmatrix} \otimes \ldots \otimes \begin{vmatrix} v_1 & \ldots & v_1 \\ \vdots & & \vdots \\ v_{\mu_{\lambda_1}} & \cdots & v_{\mu_{\lambda_1}} \end{vmatrix}$$

with one determinant-like component for each column of T_0, and where μ is the conjugate partition to λ. For example, suppose $\lambda = (2,2)$. Then $T_0 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$, and $\phi_\lambda \in T^4(V_{\mathbf{Z}})$ is given by

$$\begin{aligned} \phi_\lambda &= \begin{vmatrix} v_1 & v_1 \\ v_2 & v_2 \end{vmatrix} \otimes \begin{vmatrix} v_1 & v_1 \\ v_2 & v_2 \end{vmatrix} \\ &= (1 - (1\ 2) - (3\ 4) + (1\ 2)(3\ 4))\ t_{T_0} \\ &= v_1v_2v_1v_2 - v_2v_1v_1v_2 - v_1v_2v_2v_1 + v_2v_1v_2v_1. \end{aligned}$$

There is a map from $V_{\mathbf{Z}}$ to V sending $v \in V_{\mathbf{Z}}$ to $1 \otimes v \in V$. We denote the image of ϕ_λ under this map by $\overline{\phi}_\lambda$. Carter and Lusztig in [1] show that $V_{\lambda,\mathbf{k}}$ is a cyclic G-module generated by $\overline{\phi}_\lambda$, using the Lie algebra of G and its hyperalgebra.

2.4 Roots and the Lie Algebra.

In this section we define certain subgroups of G in terms of matrices, and certain maps of matrices into the field $\mathbf{k}$. If $\mathbf{k}$ is algebraically closed then these definitions coincide with the general theory of reductive simply connected linear algebraic groups described in Section 1.3. However, these definitions give analogous subgroups and maps in the case where $\mathbf{k}$ is a general infinite field. We will continue to use the terminology of roots and weights in this more general setting.

Denote the set $\{1,\ldots,n\}$ by I. For $i \in I$ let $e_i : S \to \mathbf{k}$ be the weight given by

$$e_i \begin{pmatrix} s_1 & & & & 0 \\ & \ddots & & & \\ & & s_i & & \\ & & & \ddots & \\ 0 & & & & s_n \end{pmatrix} \mapsto s_i.$$

Recall that the weights of S form an abelian group via

$$(\chi_1 + \chi_2)(s) = \chi_1(s)\chi_2(s) \qquad \chi_1, \chi_2 \in X(S).$$

The set of roots Φ is given by

$$\Phi = \{e_i - e_j; i, j \in I \text{ with } i \neq j\}.$$

Let $\Pi \subset \Phi$ be the subset given by

$$\Pi = \{e_i - e_{i+1}\ ;\ 1 \le i < n\}.$$

Then Π forms a set of simple roots in Φ relative to which the sets of positive and negative roots are given by

$$\begin{aligned} \Phi^+ &= \{e_i - e_j : i, j \in I \text{ and } i < j\} \\ \Phi^- &= \{e_j - e_i : i, j \in I \text{ and } i < j\}. \end{aligned}$$

Let $i, j \in I$ such that $i \neq j$. Then the root subgroup $X_{(e_i - e_j)}$ is given by

$$X_{(e_i-e_j)} = \{I_n + tE_{i,j} \ ; \ t \in \mathbf{k}\}$$

where I_n denotes the identity matrix and $E_{i,j}$ denotes the matrix with $(i,j)^{\text{th}}$ coefficient equal to 1 and all other coefficients equal to 0.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra. Then $\mathfrak{g} = M_n(\mathbf{k})$, the set of all $n \times n$ matrices over $\mathbf{k}$, with the Lie bracket given by

$$[a, b] = ab - ba$$

for all $a, b \in M_n(\mathbf{k})$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the subset of diagonal matrices, and if $\alpha = e_i - e_j$ where $i, j \in I$ and $i \neq j$ then write $e_\alpha = E_{i,j}$. We have the following decomposition of $\mathfrak{g}$ into a direct sum of the abelian subalgebra $\mathfrak{h}$ and 1-dimensional subspaces

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Phi} \mathbf{k}e_\alpha .$$

If $\mathbf{k} = \mathbf{C}$ then $\mathfrak{h}$ is a Cartan subalgebra and the above is a Cartan decomposition.

Recall that there is an embedding of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$. We will denote by $e_{i,j} \in \mathcal{U}(\mathfrak{g})$ the image in $\mathcal{U}(\mathfrak{g})$ of $E_{i,j} \in \mathfrak{g}$ under this embedding. We also write $f_{i,j} = e_{j,i}$. If we write $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ for the set of simple roots, then define $h_i \in \mathcal{U}(\mathfrak{g})$ by

$$h_i = h_{\alpha_i} = e_{i,i} - e_{j,j}.$$

Let $\mathcal{U}_\mathbf{Z}$ be the Kostant $\mathbf{Z}-form$ of $\mathcal{U}(\mathfrak{g})$ with basis consisting of elements

$$\left(\prod_{\substack{i,j\in I\\ i<j}} \frac{f_{i,j}^{b_{i,j}}}{b_{i,j}!}\right)\left(\prod_{i=1}^{n}\binom{h_i}{c_i}\right)\left(\prod_{\substack{i,j\in I\\ i<j}} \frac{e_{i,j}^{a_{i,j}}}{a_{i,j}!}\right),$$

where the $a_{i,j}$, $b_{i,j}$ and c_i are non-negative integers and where the products are taken in lexicographic order, that is

$$(1,2) < (1,3) < \ldots < (1,n) < (2,3) < (2,4) < \ldots < (n-1,n).$$

Let $\mathcal{U}_\mathbf{k} = \mathbf{k} \otimes_\mathbf{k} \mathcal{U}_\mathbf{Z}$ be the hyperalgebra of G. Then $V_{\lambda,\mathbf{k}}$ is a $\mathcal{U}_\mathbf{k}$-module and is generated under $\mathcal{U}_\mathbf{k}$ by its highest weight vector $\overline{\phi}_\lambda$. In fact if $\mathcal{U}(\mathfrak{n}^-)_\mathbf{Z}$ is the $\mathbf{Z}$-span of elements of the form

$$\prod_{\substack{i,j\in I\\ i<j}} \frac{f_{i,j}^{b_{i,j}}}{b_{i,j}!},$$

and $\mathcal{U}(\mathfrak{n}^-)_\mathbf{k} = \mathbf{k} \otimes_\mathbf{Z} \mathcal{U}(\mathfrak{n}^-)_\mathbf{Z}$ then $V_{\lambda,\mathbf{k}} = \mathcal{U}(\mathfrak{n}^-)_\mathbf{k}\overline{\phi}_\lambda$. (See Carter and Lusztig [1], 217-220.)

In the case of $GL_n(\mathbf{k})$ Weyl's dimension formula with respect to λ is equal to the number of semistandard λ-tableaux with entries from $\{1,\ldots,n\}$. Carter and Lusztig [1] construct a basis for $V_{\lambda,\mathbf{k}}$ indexed by the set of semistandard λ-tableaux with entries from $\{1,\ldots,n\}$ which we shall describe. For the rest of this chapter assume that all tableaux have entries from $\{1,\ldots,n\}$.

Let T be a λ-tableau and let μ be the conjugate partition to λ. We denote the entries of the j^{th} column of T by $t_1^j,\ldots,t_{\mu_j}^j$ for all $j \in \{1,\ldots,\lambda_1\}$. Let $\alpha \in \mathbf{Z}S_r$ be the element given by $\alpha = \sum_{\sigma\in C(\lambda)} \operatorname{sign}(\sigma)\sigma$.

2.4.1 Definition. For every λ-tableau T define an element $\phi_T \in T^r(V)$ by

$$\phi_T = \alpha\bar{t}_T = 1_{\mathbf{k}}\otimes_{\mathbf{Z}} \begin{vmatrix} v_{t_1^1} & v_{t_1^1} & v_{t_1^1} & \ldots & v_{t_1^1} \\ v_{t_2^1} & v_{t_2^1} & v_{t_2^1} & \ldots & v_{t_2^1} \\ v_{t_3^1} & v_{t_3^1} & v_{t_3^1} & \ldots & v_{t_3^1} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{t_{\mu_1}^1} & v_{t_{\mu_1}^1} & v_{t_{\mu_1}^1} & \cdots & v_{t_{\mu_1}^1} \end{vmatrix} \otimes \begin{vmatrix} v_{t_1^2} & v_{t_1^2} & \ldots & v_{t_1^2} \\ v_{t_2^2} & v_{t_2^2} & \ldots & v_{t_2^2} \\ \vdots & \vdots & & \vdots \\ v_{t_{\mu_2}^2} & v_{t_{\mu_2}^2} & \cdots & v_{t_{\mu_2}^2} \end{vmatrix} \otimes\ldots\otimes \begin{vmatrix} v_{t_1^{\lambda_1}} & \ldots & v_{t_1^{\lambda_1}} \\ \vdots & & \vdots \\ v_{t_{\mu_{\lambda_1}}^{\lambda_1}} & \cdots & v_{t_{\mu_{\lambda_1}}^{\lambda_1}} \end{vmatrix}.$$

For example, when $\lambda = (2,2)$ and $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ then $\phi_T \in T^4(V)$ is the tensor given by

$$\phi_T = \begin{vmatrix} v_1 & v_1 \\ v_2 & v_2 \end{vmatrix}\begin{vmatrix} v_3 & v_3 \\ v_4 & v_4 \end{vmatrix} = v_1v_2v_3v_4 - v_2v_1v_3v_4 - v_1v_2v_4v_3 + v_2v_1v_4v_3$$

writing $v_{i_1}v_{i_2}v_{i_3}v_{i_4}$ instead of $v_{i_1}\otimes v_{i_2}\otimes v_{i_3}\otimes v_{i_4}$ and omitting the prefix $1\otimes$. Note that $\phi_{T_0} = \overline{\phi}_\lambda$ where T_0 is the basic λ-tableau. For a λ-tableau T let $\psi_T \in T^r(V)$ be the element given by

$$\psi_T = \sum_{T'} \phi_{T'}$$

where the sum is over all λ-tableaux T' which are row permutations of T.

2.4.2 Lemma.

Let T be a semistandard λ-tableau. For all $i,j \in I$ with $i<j$ let $b_{i,j}$ equal the number of entries which are equal to j in the squares in the i^{th} row of T. Then

$$1\otimes\left(\prod_{\substack{i,j\in I\\ i<j}} \frac{f_{i,j}^{b_{i,j}}}{b_{i,j}!}\right)(\overline{\phi}_\lambda) = \psi_T$$

where the product is taken in lexicographic order.

Proof. See Carter and Lusztig [1] (46) p. 217.

□

Hence $\psi_T \in V_{\lambda,\mathbf{k}}$ for all semistandard λ-tableaux T.

2.4.3 Theorem. *The set*

$$\{\psi_T \ ; \ T \text{ is a semistandard } \lambda\text{-tableau } \}$$

forms a basis for $V_{\lambda,\mathbf{k}}$.

Proof. See Carter and Lusztig [1] (47) p. 218.

□

Over the complex numbers the module $V_{\lambda,\mathbf{C}}$ is irreducible, but when the field $\mathbf{k}$ has non-zero characteristic $V_{\lambda,\mathbf{k}}$ is not in general irreducible. However $V_{\lambda,\mathbf{k}}$ has a unique maximal submodule $M_{\lambda,\mathbf{k}}$ and the factor module

$$F_{\lambda,\mathbf{k}} \cong V_{\lambda,\mathbf{k}}/M_{\lambda,\mathbf{k}}$$

is an irreducible polynomial G-module of highest weight λ. (See Carter and Lusztig [1], p226.)

2.5 Schur Modules.

Recall the definition, for all $i,j \in \{1,\ldots,n\}$, of the coefficient function $c_{i,j} : G \to \mathbf{k}$ which is the map sending g to its $(i,j)^{\text{th}}$ coefficient $g_{i,j}$. Denote by $A_{\mathbf{k}}(n)$ the space of polynomials over $\mathbf{k}$ in the $c_{i,j}$ for all i,j, and let $A_{\mathbf{k}}(n,r)$ be the subspace of homogeneous polynomials of degree r. $A_{\mathbf{k}}(n)$ forms a left G-module under the action

$$g.c(x) = c(xg) \qquad c \in A_{\mathbf{k}}(n) \text{ and } x,g \in G.$$

We now describe another way of constructing G-modules, this time as subspaces of $A_{\mathbf{k}}(n)$. The results in this section are from Green [1].

2.5.1 Definition. The induced module $I_{\lambda,\mathbf{k}}$ corresponding to λ is given by

$$I_{\lambda,\mathbf{k}} = \{ \text{ polynomial } f : G \to \mathbf{k}; f(bg) = \lambda(b)f(g) \text{ for all } b \in B^-, g \in G\}.$$

This is a G-module under the action

$$(g.f)(x) = f(xg)$$

for all $g, x \in G$ and all $f \in I_{\lambda,\mathbf{k}}$.

$I_{\lambda,\mathbf{k}}$ is a polynomial G-module of highest weight λ and when $\mathbf{k} = \mathbf{C}$ it is irreducible. In general $I_{\lambda,\mathbf{k}}$ is not irreducible. Green [1] uses a proof from James [1] to show that $I_{\lambda,\mathbf{k}} \subset A_{\mathbf{k}}(n,r)$ and has dimension given by Weyl's dimension formula with respect to λ. We shall briefly describe the construction of $I_{\lambda,\mathbf{k}}$ in terms of the elements of $A_{\mathbf{k}}(n,r)$.

2.5.2 Definition. Let $I = \{1,\ldots,n\}$ and let $I(n,r)$ be the set of r-tuples with entries from $\{1,..,n\}$. For any $\alpha,\beta \in I(n,r)$ with $\alpha = (\alpha_1,\ldots,\alpha_r)$ and $\beta = (\beta_1,\ldots,\beta_r)$ we define an element $c_{\alpha,\beta} \in A_{\mathbf{k}}(n,r)$ by

$$c_{\alpha,\beta} = c_{\alpha_1,\beta_1} c_{\alpha_2,\beta_2} \ldots c_{\alpha_r,\beta_r}.$$

We define a left action of S_r on $A_{\mathbf{k}}(n,r)$ to be the linear extension of

$$\sigma.c_{\alpha,\beta} = c_{\alpha_1,\beta_{\sigma(1)}} c_{\alpha_2,\beta_{\sigma(2)}} \ldots c_{\alpha_r,\beta_{\sigma(r)}} \qquad \sigma \in S_r \text{ and } \alpha,\beta \in I(n,r).$$

2.5.3 Definitions. Let T be a (1^m)-tableau for some $m \leq n$ and denote the entries of T by $t_1,\ldots,t_m$. So $T = \begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline \vdots \\ \hline t_m \\ \hline \end{array}$. Then $b_T \in A_{\mathbf{k}}(n)$ is defined by

$$b_T = \begin{vmatrix} c_{1,t_1} & \ldots & c_{1,t_m} \\ \vdots & & \vdots \\ c_{m,t_1} & \cdots & c_{m,t_m} \end{vmatrix} = \sum_{\sigma \in S(m)} \operatorname{sign}(\sigma) c_{1,t_{\sigma(1)}} \ldots c_{m,t_{\sigma(m)}}.$$

Let T be a λ-tableau, and $T_1,\ldots,T_{\lambda_1}$ be the columns of T. Then the *bideterminant* $b_T \in A_{\mathbf{k}}(n)$ is given by

$$b_T = b_{T_1} \ldots b_{T_{\lambda_1}}.$$

Let $l \in I(n,r)$ be the r-tuple $(1,2,\ldots,r)$. Denote the entries of the j^{th} column of T by $t_1^j,\ldots,t_{\mu_j}^j$ for each $j \in \{1,\ldots,\lambda_1\}$. Let $t \in A_{\mathbf{k}}(n,r)$ be the r-tuple

$$(t_1^1, t_2^1, \ldots, t_{\mu_1}^1, t_1^2, \ldots, t_{\mu_2}^2, \ldots, t_{\mu_{\lambda_1}}^{\lambda_1}).$$

Then the bideterminant $b_T \in A_{\mathbf{k}}(n,r)$ has the form

$$b_T = \alpha c_{l,t} = \begin{vmatrix} c_{1,t_1^1} & c_{1,t_2^1} & \ldots & c_{1,t_{\mu_1}^1} \\ c_{2,t_1^1} & c_{2,t_2^1} & \ldots & c_{2,t_{\mu_1}^1} \\ \vdots & \vdots & & \vdots \\ c_{\mu_1,t_1^1} & c_{\mu_1,t_2^1} & \cdots & c_{\mu_1,t_{\mu_1}^1} \end{vmatrix} \begin{vmatrix} c_{1,t_1^2} & c_{1,t_2^2} & \ldots & c_{1,t_{\mu_2}^2} \\ c_{2,t_1^2} & c_{2,t_2^2} & \ldots & c_{2,t_{\mu_2}^2} \\ \vdots & \vdots & & \vdots \\ c_{\mu_2,t_1^2} & c_{\mu_2,t_2^2} & \cdots & c_{\mu_2,t_{\mu_2}^2} \end{vmatrix} \cdots \begin{vmatrix} c_{1,t_1^{\lambda_1}} & \ldots & c_{1,t_{\mu_{\lambda_1}}^{\lambda_1}} \\ \vdots & & \vdots \\ c_{\mu_{\lambda_1},t_1^{\lambda_1}} & \cdots & c_{\mu_{\lambda_1},t_{\mu_{\lambda_1}}^{\lambda_1}} \end{vmatrix}$$

2.5.4 Example. Let $n = 3$, $\lambda = (2,2)$ and let $T = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array}$. Then $b_T \in A(3,4)$ is given by

$$b_T = \begin{vmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{vmatrix} \begin{vmatrix} c_{1,1} & c_{1,3} \\ c_{2,1} & c_{2,3} \end{vmatrix}$$
$$= c_{1,1}c_{2,2}c_{1,1}c_{2,3} - c_{1,2}c_{2,1}c_{1,1}c_{2,3} - c_{1,1}c_{2,2}c_{1,3}c_{2,1} + c_{1,2}c_{2,1}c_{1,3}c_{2,1}.$$

2.5.5 Definition. We define the G-submodule $D_{\lambda,\mathbf{k}} \subset A_{\mathbf{k}}(n,r)$ to be the $\mathbf{k}$-span of the bideterminants b_T for all λ-tableaux T. We call $D_{\lambda,\mathbf{k}}$ the *Schur module* for G corresponding to λ.

2.5.6 Theorem. *$D_{\lambda,\mathbf{k}}$ is a polynomial G-module of highest weight λ, with dimension given by Weyl's dimension formula, and*

$$I_{\lambda,\mathbf{k}} \cong D_{\lambda,\mathbf{k}}$$

as G-modules.

Proof. See Green [1], Chapter 4; in particular p.64.

□

2.5.7 Theorem. *The set*

$$\{b_T \ ; \ T \textit{ is a semistandard } \lambda\textit{-tableau} \ \}$$

forms a basis of $D_{\lambda,\mathbf{k}}$.

Proof. See Green [1], p. 55.

□

2.5.8 The Contravariant Form.

The transpose map $g \mapsto g^{tr}$ is an antiautomorphism of G. For any two G-modules V and W we say that a bilinear form $< \, , \, >: V \times W \to \mathbf{k}$ is *contravariant* if

$$< g.v, w > = < v, g^{\text{tr}}.w >$$

for all $v \in V$, $w \in W$and $g \in G$. Let V and W be two G-modules and denote the dual space $\mathrm{Hom}_{\mathbf{k}}(W, \mathbf{k})$ by W^*. Define a G-action on W^* by

$$g.f(w) = f(g^{\text{tr}}.w)$$

for all $g \in G$, $f \in W^*$ and $w \in W$. Under this action W^* is called the *contravariant dual* to W and is denoted by W^0. Let $< \, , \, >: V \times W \to \mathbf{k}$ be a contravariant form. There is a homomorphism of G-modules $\theta : V \to W^0$ given by

$$\theta v(w) =< v, w >$$

for all $v \in V$ and $w \in W$. This is an isomorphism if and only if $< \, , \, >$ is non-degenerate.

In Green [1] p.70 a contravariant form $< \, , \, >: V_{\lambda,\mathbf{k}} \times D_{\lambda,\mathbf{k}} \to \mathbf{k}$ is defined which is non-degenerate. Hence each of $V_{\lambda,\mathbf{k}}$ and $D_{\lambda,\mathbf{k}}$ is isomorphic to the contravariant dual of the other. Recall the unique maximal submodule $M_{\lambda,\mathbf{k}}$ of $V_{\lambda,\mathbf{k}}$ and the factor module $F_{\lambda,\mathbf{k}} = V_{\lambda,\mathbf{k}}/M_{\lambda,\mathbf{k}}$ which is an irreducible polynomial G-module of highest weight λ. It follows that $D_{\lambda,\mathbf{k}}$ has a unique minimal submodule isomorphic to $F_{\lambda,\mathbf{k}}$.

2.5.9 Theorem.. *Every irreducible polynomial G-module is isomorphic to $F_{\lambda,\mathbf{k}}$ for exactly one $\lambda \in X(S)^+$.*

Proof. See Green [1] p.45.

□

So as λ varies over all partitions of r into not more than n parts, the set $\{F_{\lambda,\mathbf{k}}\}$ forms a complete set of irreducible polynomial G-modules.

3

Symplectic Groups.

In Chapter 1 we reviewed some of the basic theory of reductive linear algebraic groups over an algebraically closed field of characteristic zero, and described some of their structure. Over an algebraically closed field the symplectic groups are simple simply connected linear algebraic groups. However, we are considering the more general case over any infinite field, and in this chapter we describe some of the structure retained in this more general setting. Let $\mathbf{k}$ denote any infinite field.

3.1 Definition of Symplectic Groups.

Let V be a vector space over an infinite field $\mathbf{k}$. A bilinear form $<,>$ is *skew-symmetric* if $< v,v >= 0$ for all $v \in V$. This immediately implies that $< u,v >= - < v,u >$ for all $u,v \in V$. A bilinear form $<,>$ is *non-degenerate* if the only $w \in V$ satisfying $< v,w >= 0$ for all $v \in V$ is $w = 0$. A space endowed with a non-degenerate skew-symmetric bilinear form is called a *symplectic space.*

Any two non-degenerate skew-symmetric bilinear forms on V are equivalent. Moreover, any finite-dimensional vector space V on which a non-degenerate skew-symmetric form is defined must have even dimension $2n$, $n \in \mathbb{N}$, and a basis $\{v_1, \ldots, v_n, v_{\bar{n}}, \ldots, v_{\bar{1}}\}$ satisfying $< v_i, v_{\bar{i}} >= - < v_{\bar{i}}, v_i >= 1$ for all $i \in \{1, \ldots, n\}$ with all other products zero. Such a basis of V is called a *symplectic basis.*

A linear mapping on a symplectic space which preserves the value of the bilinear form is called a *symplectic map*. When it is also an isomorphism it is called an *isometry*. The isometries of a symplectic space form a group, called a *symplectic group*. When V is of dimension $2n$ over $\mathbf{k}$ the symplectic group on V is denoted by $Sp_{2n}(\mathbf{k})$.

3.1.1 The group in matrix form.

Let V be a symplectic space of dimension $2n$ over $\mathbf{k}$ and let $\{v_1, \ldots, v_n, v_{\bar{n}}, \ldots, v_{\bar{1}}\}$ be a symplectic basis. Relative to this basis the form is represented by the matrix

$$A = \begin{pmatrix} & 0 & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{pmatrix}$$

where $< v_i, v_j >= (A)_{i,j}$ for all $i, j \in \{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$.

Notice that A satisfies $a_{ii} = 0$ and $a_{ij} = -a_{ji}$. Any matrix satisfying these two conditions is said to be *skew-symmetric*. In fact, relative to any basis, a skew-symmetric form specifies a skew-symmetric matrix, and *vice versa*. Hence we see that any non-singular skew-symmetric matrix must be congruent to A.

The symplectic group $Sp_{2n}(\mathbf{k})$ is the group of all isometries of V, and hence is isomorphic to the group of all $2n \times 2n$ matrices such that

$$S^T A S = A.$$

Such a matrix is called *symplectic,* and has determinant 1 (see Weyl [1] p. 166-167).

3.2 Symplectic Roots and Weights.

In this section we will define certain subgroups of $Sp_{2n}(\mathbf{k})$ in terms of matrices, and certain maps of matrices into the field $\mathbf{k}$. If $\mathbf{k}$ is algebraically closed then these definitions coincide with the general theory of reductive simply connected linear algebraic groups described in Chapter 1. However, these definitions give analogous subgroups and maps in the case where $\mathbf{k}$ is a general infinite field. We will continue to use the terminology of roots and weights in this more general setting.

Henceforth we write Sp as a shortened form of $Sp_{2n}(\mathbf{k})$. Let $I = \{1, \ldots, n\}$ and $\overline{I} = \{\overline{1}, \ldots, \overline{n}\}$. The subgroup of diagonal matrices $T \subset Sp$ forms a maximal torus in Sp. An element $t \in T$ is of the form

$$t = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & 0 & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \qquad t_i \in \mathbf{k} \setminus \{0\} \text{ for all } i \in I.$$

A *weight* of $Sp_{2n}(\mathbf{k})$ is a 1-dimensional representation of T which maps $t \in T$ to an integral polynomial in the $\{t_i; i \in I\}$. We denote the set of weights by $X(T)$. Each $\lambda \in X(T)$ can be described by an n-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i \in \mathbf{Z}$ for all $i \in I$, and

$$\lambda(t) = t_1^{\lambda_1} t_2^{\lambda_2} \ldots \, t_n^{\lambda_n} \qquad \text{for all } t \in T.$$

The set of dominant weights $X(T)^+ \subset X(T)$ consists of all weights $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

For $i \in I$ define the weight $e_i : T \to \mathbf{k}$ by

$$e_i \begin{pmatrix} t_1 & & & & & & & \\ & \ddots & & & & 0 & & \\ & & t_i & & & & & \\ & & & \ddots & & & & \\ & & & & t_n & & & \\ & & & & & t_n^{-1} & & \\ & 0 & & & & & \ddots & \\ & & & & & & & t_1^{-1} \end{pmatrix} \mapsto t_i.$$

Recall that the weights of T form an abelian group under addition defined by

$$(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t) \qquad \chi_1, \chi_2 \in X(T),\ t \in T.$$

The set of roots Φ of $Sp_{2n}(\mathbf{k})$ is given by

$$\Phi = \{\pm e_i \pm e_j,\ \pm 2e_k;\ i, j \in I,\ i < j \text{ and } k \in I\}.$$

Let $\Pi \subset \Phi$ be the subset given by

$$\Pi = \{e_i - e_{i+1};\ 1 \leq i < n\} \cup \{2e_n\}.$$

Then Π forms a set of simple roots in Φ relative to which the sets Φ^+ of positive roots and Φ^- of negative roots are given by

$$\Phi^+ = \left\{ \begin{array}{ll} x_{ij} = e_i + e_j & \\ y_{ij} = e_i - e_j \qquad ; & i, j \in I \text{ with } i < j \text{ and } k \in I \\ z_k = 2e_k & \end{array} \right\}$$

$$\Phi^- = \left\{ \begin{array}{ll} a_{ij} = -e_i - e_j & \\ b_{ij} = e_j - e_i \qquad ; & i, j \in I \text{ with } i < j \text{ and } k \in I \\ c_k = -2e_k & \end{array} \right\}$$

Notice that $a_{i,j} = -x_{i,j}$ and $b_{i,j} = -y_{i,j}$ for all i, j and $c_i = -z_i$ for all i. For $i, j \in I \cup \overline{I}$ let $E_{i,j}$ denote the matrix with 1 as the $(i,j)^{\text{th}}$ coefficient and all other coefficients 0. the root subgroups of Sp are

$$\begin{array}{ll} U_{a_{i,j}} = \{I + t(E_{\overline{i},j} + E_{\overline{j},i});\ t \in \mathbf{k}\} & U_{x_{i,j}} = \{I + t(E_{i,\overline{j}} + E_{j,\overline{i}});\ t \in \mathbf{k}\} \\ U_{b_{i,j}} = \{I + t(E_{j,i} - E_{\overline{i},\overline{j}});\ t \in \mathbf{k}\} & U_{y_{i,j}} = \{I + t(E_{i,j} - E_{\overline{j},\overline{i}});\ t \in \mathbf{k}\} \\ U_{c_k} = \{I + tE_{\overline{k},k};\ t \in \mathbf{k}\} & U_{z_k} = \{I + tE_{k,\overline{k}};\ t \in \mathbf{k}\}, \end{array}$$

for $1 \leq i < j \leq n$ and all $k \in I$.

We define $U_{sp}^- \subset Sp$ to be the subgroup generated by all the negative root subgroups of Sp; that is

$$U_{sp}^- = < U_{a_{i,j}}, U_{b_{i,j}}, U_{c_k};\ \text{for all } i, j \in I, i < j \text{ and } k \in I >$$

Define the subgroup $B^- \subset Sp$ to be the product $B^- = (U_{sp}^-)T$. Since T normalises U_{sp}^- we have $U_{sp}^- \triangleleft B^-$. Also $U_{sp}^- \cap T = 1$ and consequently B^- is a semidirect product of U_{sp}^- and T. Every element $b \in B^-$ has a unique expression $b = ut$ where $u \in U_{sp}^-$ and $t \in T$.

When $\mathbf{k}$ is algebraically closed the subgroup B^- is a Borel subgroup containing T. In general, if we order the rows and columns of the matrices in Sp according to

$$1 < 2 < \ldots < n < \overline{n} < \overline{n-1} < \ldots < \overline{1}$$

then B^- consists of all the lower triangular matrices in $Sp_{2n}(\mathbf{k})$, and U_{sp}^- of all the lower unitriangular matrices in $Sp_{2n}(\mathbf{k})$.

3.3 The Lie Algebra.

Let $\mathfrak{sp}_{2n}(\mathbf{k}) = \mathrm{Lie}(Sp_{2n}(\mathbf{k}))$. The Lie algebra $\mathfrak{sp}_{2n}(\mathbf{k})$ is isomorphic to the set of $2n \times 2n$ matrices S with entries in $\mathbf{k}$ satisfying

$$S^T A + AS = 0$$

where A is as in Section 3.1.1. Note that $\mathfrak{sp}_{2n}(\mathbf{C})$ is isomorphic to the simple Lie algebra over $\mathbf{C}$ of type C_l.

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{sp}_{2n}(\mathbf{k})$. The elements of $\mathfrak{h}$ can all be written in the form

$$\begin{pmatrix} h_1 & & & & & \\ & \ddots & & & 0 & \\ & & h_n & & & \\ & & & -h_n & & \\ & 0 & & & \ddots & \\ & & & & & -h_1 \end{pmatrix},$$

where $h_1, \ldots, h_n \in \mathbf{k}$. When $\mathbf{k} = \mathbf{C}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sp}_{2n}(\mathbf{C})$. Index the rows and columns of the matrices in $\mathfrak{sp}_{2n}(\mathbf{C})$ by $\{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$. For all $i, j \in I \cup \overline{I}$ denote the elementary matrix, with 1 in the $(i,j)^{\text{th}}$-position and 0 everywhere else, by $E_{i,j}$. There is a Cartan decomposition of $\mathfrak{sp}_{2n}(\mathbf{C})$ given by

$$\mathfrak{sp}_{2n}(\mathbf{C}) = \mathfrak{h} \bigoplus_{r \in \Phi} \mathbf{C} e_r,$$

with the vectors e_r ranging over the following root vectors.

$$\left.\begin{aligned} e_{x_{i,j}} &= E_{i,\overline{j}} + E_{j,\overline{i}}, \\ e_{a_{i,j}} &= E_{\overline{i},j} + E_{\overline{j},i}, \\ e_{y_{i,j}} &= E_{i,j} - E_{\overline{j},\overline{i}}, \\ e_{b_{i,j}} &= E_{j,i} - E_{\overline{i},\overline{j}} \end{aligned}\right\} \quad \text{for } i, j \in I \text{ and } i < j$$

and

$$\left.\begin{aligned} e_{z_i} &= E_{i,\overline{i}} \\ e_{c_i} &= E_{\overline{i},i} \end{aligned}\right\} \quad \text{for } i \in I.$$

For a general infinite field $\mathbf{k}$ it remains true that

$$\mathfrak{sp}_{2n}(\mathbf{k}) = \mathfrak{h} \oplus \mathbf{k} e_r,$$

where the e_r are the same matrices as described above.

3.4 The Symplectic Hyperalgebra.

Recall the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\} = \{y_{1,2},\ y_{2,3},\ \ldots,\ y_{n-1,n},\ z_n\}$.

3.4.1 Definition. For $i \in I$ we define an element $h_i \in \mathfrak{h}$ by

$$h_i = h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}].$$

Thus

$$h_i = \begin{cases} E_{i,i} - E_{\bar{i},\bar{i}} - E_{i+1,i+1} + E_{\overline{i+1},\overline{i+1}} & \text{for } 1 \le i < n; \\ E_{n,n} - E_{\bar{n},\bar{n}} & \text{for } i = n. \end{cases}$$

Let $\mathcal{U}^{sp}_{\mathbb{C}} = \mathcal{U}(\mathfrak{sp}_{2n}(\mathbb{C}))$ be the universal enveloping algebra of $\mathfrak{sp}_{2n}(\mathbb{C})$. Let $r \in \Phi$ and let $e_r \in \mathfrak{sp}_{2n}(\mathbb{C})$ be a root vector. We denote the corresponding element to e_r under the natural embedding of $\mathfrak{sp}_{2n}(\mathbb{C})$ in $\mathcal{U}^{sp}_{\mathbb{C}}$ by r. So, for example, the element $a_{i,j} \in \mathcal{U}^{sp}_{\mathbb{C}}$ is the image of $e_{a_{i,j}} \in \mathfrak{sp}_{2n}(\mathbb{C})$ under this embedding. Then $\mathcal{U}^{sp}_{\mathbb{C}}$ has a Kostant basis consisting of elements

$$\left(\prod_{\substack{i,j\in I\\ i<j}} \frac{a_{i,j}^{\alpha_{i,j}}}{\alpha_{i,j}!}\frac{b_{i,j}^{\beta_{i,j}}}{\beta_{i,j}!}\right)\left(\prod_{i\in I}\frac{c_i^{\gamma_i}}{\gamma_i!}\right)\left(\prod_{i\in I}\binom{h_i}{\eta_i}\right)\left(\prod_{i\in I}\frac{z_i^{\zeta_i}}{\zeta_i!}\right)\left(\prod_{\substack{i,j\in I\\ i<j}} \frac{x_{i,j}^{\chi_{i,j}}}{\chi_{i,j}!}\frac{y_{i,j}^{\varphi_{i,j}}}{\varphi_{i,j}!}\right)$$

where $\alpha_{i,j}$, $\beta_{i,j}$, γ_i, η_i, ζ_i, $\chi_{i,j}$ and $\varphi_{i,j}$ are all non-negative integers, and where each product is taken in lexicographic order; that is

$$(1,2) < (1,3) < \ldots < (1,n) < (2,3) < \ldots < (n-1,n).$$

The $\mathbb{Z}$-span of all such basis elements is the Kostant $\mathbb{Z}$-form $\mathcal{U}^{sp}_{\mathbb{Z}}$ of $\mathcal{U}^{sp}_{\mathbb{C}}$, and the symplectic hyperalgebra $\mathcal{U}^{sp}_{\mathbf{k}}$ is given by

$$\mathcal{U}^{sp}_{\mathbf{k}} = \mathbf{k} \otimes_{\mathbb{Z}} \mathcal{U}^{sp}_{\mathbb{Z}}.$$

So a basis for $\mathcal{U}^{sp}_{\mathbf{k}}$ consists of elements of the above form tensored with the identity element $1 \in \mathbf{k}$. For simplicity, for any $u \in \mathcal{U}^{sp}_{\mathbb{Z}}$ we shall write u in place of $1 \otimes u \in \mathcal{U}^{sp}_{\mathbf{k}}$.

3.4.2 Definition. Let $\mathcal{U}^-_{Sp,\mathbb{Z}}, \mathcal{U}^0_{Sp,\mathbb{Z}}, \mathcal{U}^+_{Sp,\mathbb{Z}} \subset \mathcal{U}^{sp}_{\mathbb{Z}}$ be given by

$$\mathcal{U}^-_{Sp,\mathbb{Z}} = \mathbb{Z}\text{-span of } \left\{\left(\prod_{\substack{i,j\in I\\ i<j}} \frac{a_{i,j}^{\alpha_{i,j}}}{\alpha_{i,j}!}\frac{b_{i,j}^{\beta_{i,j}}}{\beta_{i,j}!}\right)\left(\prod_{i\in I}\frac{c_i^{\gamma_i}}{\gamma_i!}\right) \text{ for all } \alpha_{i,j}, \beta_{i,j}, \gamma_i \in \mathbb{N}\cup\{0\}\right\}$$

$$\mathcal{U}^0_{Sp,\mathbb{Z}} = \mathbb{Z}\text{-span of } \left\{\prod_{i\in I}\binom{h_i}{\eta_i} \text{ for all } \eta_i \in \mathbb{N}\cup\{0\}\right\}$$

$$\mathcal{U}^+_{Sp,\mathbb{Z}} = \mathbb{Z}\text{-span of } \left\{\left(\prod_{i\in I}\frac{z_i^{\zeta_i}}{\zeta_i!}\right)\left(\prod_{\substack{i,j\in I\\ i<j}} \frac{x_{i,j}^{\chi_{i,j}}}{\chi_{i,j}!}\frac{y_{i,j}^{\varphi_{i,j}}}{\varphi_{i,j}!}\right) \text{ for all } \chi_{i,j}, \varphi_{i,j}, \zeta_i \in \mathbb{N}\cup\{0\}\right\}$$

It is clear that $\mathcal{U}^{sp}_{\mathbf{Z}} = \mathcal{U}^{-}_{Sp,\mathbf{Z}}\mathcal{U}^{0}_{Sp,\mathbf{Z}}\mathcal{U}^{+}_{Sp,\mathbf{Z}}$.

3.5 Polynomial Functions on $Sp_{2n}(\mathbf{k})$.

Let $I = \{1,\ldots,n\}$ and $\overline{I} = \{\overline{1},\ldots,\overline{n}\}$ as before. Recall the coefficient functions $c_{i,j} : GL_{2n}(\mathbf{k}) \to \mathbf{k}$ which map a matrix to its $(i,j)^{\text{th}}$-coefficient, for any $i,j \in I \cup \overline{I}$.

These maps are linearly independent over $\mathbf{k}$. However, when their action is restricted to $Sp_{2n}(\mathbf{k}) \subset GL_{2n}(\mathbf{k})$ this is no longer true, and they satisfy certain relations. Let $A_{\mathbf{k}}(\overline{n})$ be the algebra of all polynomials in the coefficient functions $c_{i,j}$ for all $i,j \in I \cup \overline{I}$. We wish to find a set of polynomials in $A_{\mathbf{k}}(\overline{n})$ such that the set of matrices in $GL_{2n}(\mathbf{k})$ on which they are all zero is $Sp_{2n}(\mathbf{k})$.

Let

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

be a $2n \times 2n$ matrix where $S_{11},\ldots,S_{22}$ are of size $n \times n$. Reordering the basis for V to be in the order $v_1, v_2, \ldots, v_n, v_{\overline{1}}, v_{\overline{2}}, \ldots, v_{\overline{n}}$, the condition for S to be in $Sp_{2n}(\mathbf{k})$ is $S^T A' S = A'$, where

$$A' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

By multiplying out the matrices this condition is equivalent to the conditions

$$\begin{aligned} S_{11}S_{12}^T - S_{12}S_{11}^T &= 0 \\ S_{21}S_{22}^T - S_{22}S_{21}^T &= 0 \\ S_{11}S_{22}^T - S_{12}S_{21}^T &= I_n. \end{aligned}$$

By comparing the $(i,j)^{\text{th}}$-coefficients on each side of the equality and by using the rules for matrix multiplication these are equivalent to

$$\sum_{k=1}^{n}(c_{i,k}c_{j,\overline{k}} - c_{i,\overline{k}}c_{j,k}) = 0$$

for all $i,j \in I \cup \overline{I}$ and $\{i,j\} \neq \{\mu,\overline{\mu}\}$ for any μ, and

$$\sum_{k=1}^{n}(c_{i,k}c_{\overline{i},\overline{k}} - c_{i,\overline{k}}c_{\overline{i},k}) = 1$$

for all $i \in I$.

3.5.1 Definition. The quadratic form $Q_{i,j}$ is given by

$$Q_{i,j} = \sum_{k=1}^{n} \begin{vmatrix} c_{i,k} & c_{i,\overline{k}} \\ c_{j,k} & c_{j,\overline{k}} \end{vmatrix},$$

for all $i, j \in I \cup \bar{I}$.

So the above calculation shows the following.

3.5.2 Lemma. *A system of defining relations for $Sp_{2n}(\mathbf{k})$ is given by*

$$
\begin{aligned}
Q_{i,j} &= 0 \\
Q_{i,\bar{j}} &= 0 \\
Q_{\bar{i},j} &= 0 \\
Q_{\bar{i},\bar{j}} &= 0 \\
Q_{k,\bar{k}} &= 1
\end{aligned}
$$

for all $i, j, k \in \{1, \ldots, n\}$ such that $i < j$.

These are called the *quadratic relations.* The set

$$\{Q_{i,j},\ Q_{i,\bar{j}},\ Q_{\bar{i},j},\ Q_{\bar{i},\bar{j}},\ Q_{i,\bar{i}} - 1\}$$

generates an ideal $\mathfrak{J} \subset A_{\mathbf{k}}(\overline{n})$. Any $2n \times 2n$ matrix S is in $Sp_{2n}(\mathbf{k})$ if and only if all the functions in $\mathfrak{J}$ vanish on S. Hence the set of functions vanishing on $Sp_{2n}(\mathbf{k})$ is the radical ideal $\sqrt{\mathfrak{J}}$ of $\mathfrak{J}$.

3.5.3 Definitions. For any $i, j \in I \cup \bar{I}$, let $d_{i,j}$ denote the restriction of $c_{i,j}$ to the symplectic group. Define $A_{\mathbf{k}}^{sp}(\overline{n})$ by

$$A_{\mathbf{k}}^{sp}(\overline{n}) = A_{\mathbf{k}}(\overline{n})/\sqrt{\mathfrak{J}},$$

and let $\pi : A_{\mathbf{k}}(\overline{n}) \longrightarrow A_{\mathbf{k}}^{sp}(\overline{n})$ be the canonical map. Then $\pi(c_{i,j}) = d_{i,j}$ and $A_{\mathbf{k}}^{sp}(\overline{n})$ is the algebra over $\mathbf{k}$ of polynomials in the functions $d_{i,j}$ for all $i, j \in I \cup \bar{I}$.

Let $A_{\mathbf{k}}^{sp}(\overline{n}, r) \subset A_{\mathbf{k}}^{sp}(\overline{n})$ denote the polynomials in the $d_{i,j}$ which are homogeneous of degree r.

3.6 The Symplectic Relations.

There is a right $Sp_{2n}(\mathbf{k})$-action on $A_{\mathbf{k}}^{sp}(\overline{n})$ given by

$$(f \circ s)(g) = f(sg)$$

for all $s, g \in Sp_{2n}(\mathbf{k})$, and all $f \in A_{\mathbf{k}}^{sp}(\overline{n})$.

We are interested in the polynomials which are invariant under the right action by elements in U_{sp}^{-}. Any $f \in A_{\mathbf{k}}^{sp}(\overline{n})$ is a sum of monomials, and it will be sufficient to determine the invariant monomials. Let $d \in A_{\mathbf{k}}^{sp}(\overline{n}, r)$ be a monomial of degree r, and let

$I(\overline{n},r)$ be the set of r-tuples $\alpha = (\alpha_1,\ldots,\alpha_r)$ with each $\alpha_i \in I \cup \overline{I}$. Then we can write $d = d_{\alpha_1,\beta_1}d_{\alpha_2,\beta_2}\ldots d_{\alpha_r,\beta_r}$ for some α, $\beta \in I(\overline{n},r)$, and denote d by $d_{\alpha,\beta}$.

3.6.1 Definitions. For $i,j \in I$ with $i<j$ and all $v \in \mathbf{Z}$ such that $v \geq 0$ define a symplectic function

$$\psi_v^{a_{ij}} : A_{\mathbf{k}}^{sp}(\overline{n},r) \to A_{\mathbf{k}}^{sp}(\overline{n},r)$$

to be the linear extension of

$$\psi_v^{a_{ij}}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\gamma,\beta} \quad \alpha,\beta \in I(\overline{n},r)$$

summed over all $\gamma \in I(\overline{n},r)$ obtained from α by replacing some entries $\overline{i}$ by j and some entries $\overline{j}$ by i, such that the total number of replacements is v. If the total number of $\overline{i}$'s and $\overline{j}$'s in α is less than v then $\psi_v^{a_{ij}}(d_{\alpha,\beta}) = 0$.

For $i,j \in I$ with $i<j$ and all $v,w \in \mathbf{Z}$ such that $v,w \geq 0$ define a symplectic function

$$\psi_{v,w}^{b_{ij}} : A_{\mathbf{k}}^{sp}(\overline{n},r) \to A_{\mathbf{k}}^{sp}(\overline{n},r)$$

to be the linear extension of

$$\psi_{v,w}^{b_{ij}}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\gamma,\beta} \quad \alpha,\beta \in I(\overline{n},r)$$

summed over all $\gamma \in I(\overline{n},r)$ obtained from α by replacing v of the entries $\overline{i}$ by $\overline{j}$ and w of the entries j by i. If there are less than v $\overline{i}$'s or less than w j's in α then $\psi_{v,w}^{b_{ij}}(d_{\alpha,\beta}) = 0$.

For $i \in I$ and all $v \in \mathbf{Z}$ such that $v \geq 0$ define a symplectic function

$$\psi_v^{c_i} : A_{\mathbf{k}}^{sp}(\overline{n},r) \to A_{\mathbf{k}}^{sp}(\overline{n},r)$$

to be the linear extension of

$$\psi_v^{c_i}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\gamma,\beta} \quad \alpha,\beta \in I(\overline{n},r)$$

summed over all $\gamma \in I(\overline{n},r)$ obtained from α by replacing v of the entries $\overline{i}$ by i. If there are less than v $\overline{i}$'s in α then $\psi_v^{c_i}(d_{\alpha,\beta}) = 0$.

3.6.2 Examples. Consider $Sp_4(\mathbf{k})$. Let $\alpha = (\overline{1},\overline{1},2,\overline{2})$, $\beta = (1,2,\overline{2},1)$ and let

$$d = d_{\alpha,\beta} = d_{\overline{1},1}d_{\overline{1},2}d_{2,\overline{2}}d_{\overline{2},1}.$$

Then

$$\psi_2^{a_{12}}(d) = d_{21}d_{22}d_{2\overline{2}}d_{\overline{2}1} + d_{21}d_{\overline{1}2}d_{2\overline{2}}d_{11} + d_{\overline{1}1}d_{22}d_{2\overline{2}}d_{11},$$

where the γ's obtained are $(2,2,2,\overline{2})$, $(2,\overline{1},2,1)$ and $(\overline{1},2,2,1)$.

Also

$$\psi_{1,1}^{b_{12}}(d) = d_{\bar{2}1}d_{\bar{1}2}d_{1\bar{2}}d_{\bar{2}1} + d_{\bar{1}1}d_{\bar{2}2}d_{1\bar{2}}d_{\bar{2}1}$$
$$\psi_{1}^{c_1}(d) = d_{11}d_{\bar{1}2}d_{2\bar{2}}d_{\bar{2}1} + d_{\bar{1}1}d_{12}d_{2\bar{2}}d_{\bar{2}1}$$
$$\psi_{1}^{c_2}(d) = d_{\bar{1}1}d_{\bar{1}2}d_{2\bar{2}}d_{21}$$

3.6.3 Definition. Let $f \in A_{\mathbf{k}}^{sp}(\overline{n})$. Then f satisfies the *symplectic relations* if, for all $i, j \in I$ such that $i < j$ and all $k \in I$, f satisfies:-

(i) $\psi_v^{a_{ij}}(f) = 0$ for all $v \in \mathbb{N}$;
(ii) $\sum_{v=1}^{N} \psi_{v,N-v}^{b_{ij}}(f) = 0$ for all $N \in \mathbb{N}$;
(iii) $\psi_v^{c_k}(f) = 0$ for all $v \in \mathbb{N}$.

3.6.4 Lemma. *Let $f \in A^{sp}(\overline{n})$. Then f satisfies $f \circ u = f$ for all $u \in U_{sp}^-$ if and only if f satisfies the symplectic relations.*

Proof. First consider the action of U_{sp}^- on monomials in $A^{sp}(\overline{n})$. This action can be studied via the actions of the root subgroups. Let $d_{\alpha,\beta} \in A^{sp}(\overline{n})$ be a monomial with $\alpha, \beta \in I(\overline{n}, r)$.

Let $i, j \in I$ such that $i < j$. An element $u \in U_{a_{ij}}$ is of the form

$$u = I + t(E_{\bar{i}j} + E_{\bar{j}i})$$

for some $t \in \mathbf{k}$. Now $(d_{\alpha,\beta} \circ u)(g) = d_{\alpha,\beta}(ug)$ for any $g \in Sp_{2n}(\mathbf{k})$ and by the rules of matrix multiplication we see that

$$d_{\alpha,\beta} \circ u = \sum_{\gamma \in I(\overline{n},r)} d_{\alpha,\gamma}(u) d_{\gamma,\beta}.$$

However $d_{\alpha,\gamma}(u) = 0$ unless

$$(\alpha_\rho, \gamma_\rho) \in \{(1,1), \ldots, (n,n), (\bar{1},\bar{1}), \ldots, (\bar{n},\bar{n}), (\bar{i},j), (\bar{j},i)\}$$

for all $\rho \in \{1, \ldots, r\}$. If this is so then $d_{\alpha,\gamma}(u) = t^v$ where v is the number of ρ such that $(\alpha_\rho, \gamma_\rho) \in \{(\bar{i},j), (\bar{j},i)\}$.

Hence

$$d_{\alpha,\beta} \circ u = \sum_{v=0}^{r} t^v \psi_v^{a_{ij}}(d_{\alpha,\beta}). \tag{A}$$

Let $u \in U_{b_{ij}}$. Then u is of the form

$$u = I + t(E_{ji} - E_{\bar{i}\bar{j}})$$

for some $t \in \mathbf{k}$ and

$$d_{\alpha,\beta} \circ u = \sum_{\gamma \in I(\overline{n},r)} d_{\alpha,\gamma}(u) d_{\gamma,\beta}.$$

As above, we have that $d_{\alpha,\gamma}(u) = 0$ unless

$$(\alpha_\rho, \gamma_\rho) \in \{(1,1), \ldots, (n,n), (\bar{1},\bar{1}), \ldots, (\bar{n},\bar{n}), (j,i), (\bar{i},\bar{j})\}$$

for all $\rho \in \{1, \ldots, r\}$. If this is so then $d_{\alpha,\gamma}(u) = (-1)^w t^{v+w}$, where v is the number of ρ such that $(\alpha_\rho, \gamma_\rho) = (j,i)$. and w is the number of ρ such that $(\alpha_\rho, \gamma_\rho) = (\bar{i},\bar{j})$ So we have shown that

$$d_{\alpha,\beta} \circ u = \sum_{v=0}^{r} \sum_{w=0}^{r} (-1)^w t^{v+w} \psi_{v,w}^{b_{ij}}(d_{\alpha,\beta}). \tag{B}$$

Finally, let $u \in U_{c_i}$ for any $i \in I$. Then $u = I + tE_{\bar{i}i}$ for some $t \in \mathbf{k}$, and

$$d_{\alpha,\beta} \circ u = \sum_{\gamma \in I(\bar{n},r)} d_{\alpha,\gamma}(u) d_{\gamma,\beta}.$$

Now $d_{\alpha,\gamma}(u) = 0$ unless

$$(\alpha_\rho, \gamma_\rho) \in \{(1,1), \ldots, (n,n), (\bar{1},\bar{1}), \ldots, (\bar{n},\bar{n}), (\bar{i},i)\}$$

for all $\rho \in \{1, \ldots, r\}$. If this is so $d_{\alpha,\gamma}(u) = t^v$, where v is the number of ρ such that $(\alpha_\rho, \gamma_\rho) = (\bar{i}, i)$. Hence

$$d_{\alpha,\beta} \circ u = \sum_{v=0}^{r} t^v \psi_v^{c_i}(d_{\alpha,\beta}). \tag{C}$$

Let $f \in A_{\mathbf{k}}(\bar{n})$ and assume that conditions (i), (ii) and (iii) are satisfied by f. When $v = w = 0$ we have

$$\psi_v^{a_{ij}}(f) = f$$
$$\psi_{v,w}^{b_{ij}}(f) = f$$
$$\psi_v^{c_k}(f) = f$$

for all $i, j, k \in I$ such that $i < j$. By equations (A), (B) and (C) we see that for all u in the root subgroups $U_{a_{ij}}$, $U_{b_{ij}}$ and U_{c_i} we have that

$$f \circ u = f$$

and since these subgroups generate U_{sp}^- then

$$f \circ u = f \text{ for all } u \in U_{sp}^-.$$

Conversely, let $f \in A_{\mathbf{k}}(\bar{n})$ be homogeneous of degree r and satisfy $f \circ u = f$ for all $u \in U_{sp}^-$. Then for any $i, j \in I$ with $i < j$ we have

$$f \circ u = f$$

for all $u \in U_{a_{ij}}$. By equation (A) $f \circ u = \sum_{v=0}^{r} t^v \psi_v^{a_{ij}}(f)$. When $v = 0$ then $\psi_v^{a_{ij}}(f) = f$ and so

$$\sum_{v=1}^{r} t^v \psi_v^{a_{ij}}(f) = 0. \tag{D}$$

Let $A_v = \psi_v^{a_{i,j}}(f)$ for $v \in \{1, \ldots, r\}$. Then we have an equation

$$A_1 t + A_2 t^2 + \ldots + A_r t^r = 0.$$

Since $\mathbf{k}$ is an infinite field we can choose r non-zero elements $\zeta_1, \ldots, \zeta_r \in \mathbf{k}$ such that $\zeta_i - \zeta_j \neq 0$ for all $i, j \in \{1, \ldots, r\}$ with $i < j$. Then

$$\begin{pmatrix} \zeta_1^r & \zeta_1^{r-1} & \ldots & \zeta_1 \\ \zeta_2^r & \zeta_2^{r-1} & \ldots & \zeta_2 \\ \vdots & \vdots & & \vdots \\ \zeta_r^r & \zeta_r^{r-1} & \cdots & \zeta_r \end{pmatrix} \begin{pmatrix} A_r \\ A_{r-1} \\ \vdots \\ A_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now

$$\det \begin{pmatrix} \zeta_1^r & \zeta_1^{r-1} & \ldots & \zeta_1 \\ \zeta_2^r & \zeta_2^{r-1} & \ldots & \zeta_2 \\ \vdots & \vdots & & \vdots \\ \zeta_r^r & \zeta_r^{r-1} & \cdots & \zeta_r \end{pmatrix} = \prod_{i=1}^{r} \zeta_i \cdot \prod_{1 \leq i < j \leq r} (\zeta_i - \zeta_j) \neq 0,$$

and the matrix is invertible. By multiplying both sides of the matrix equation by the inverse we obtain $A_1 = A_2 = \ldots = A_r = 0$. Hence the only solution to equation (D) is $\psi_v^{a_{ij}}(f) = 0$ for each $v \in \{1, \ldots, r\}$. By definition $\psi_v^{a_{ij}}(f) = 0$ whenever $v > r$.

Similarly, we have $\psi_v^{c_i}(f) = 0$ for all $i \in I$ and $v > 0$, and $\sum_{v=1}^{N} (-1)^{N-v} \psi_{v,N-v}^{b_{ij}}(f) = 0$ for all $i, j \in I$ such that $i < j$, and for all $N > 0$. So f satisfies the symplectic relations.

□

3.7 Functions Arising from the Left Action.

We are interested in the action of certain elements of $Sp_{2n}(\mathbf{k})$ on the left of $A_{\mathbf{k}}^{sp}(\overline{n})$. These will be used to find a basis for the symplectic Schur module in Chapter 6.

3.7.1 Definition. Let $i, j, k \in I$ with $i < j$ and let $t \in \mathbf{k}$. Define the elements $x_{i,j}(t), y_{i,j}(t), z_k(t) \in Sp_{2n}(\mathbf{k})$ by

$$\begin{aligned} x_{i,j}(t) &= I + t(E_{i,\overline{j}} + E_{j,\overline{i}}) \\ y_{i,j}(t) &= I + t(E_{i,j} - E_{\overline{j},\overline{i}}) \\ z_k(t) &= I + tE_{k,\overline{k}}. \end{aligned}$$

These are all upper unitriangular matrices in $Sp_{2n}(\mathbf{k})$ and we are interested in their left actions on $A_{\mathbf{k}}^{sp}(\overline{n})$.

3.7.2 Definition. For $i, j \in I$ with $i < j$ and all $v \in \mathbf{Z}$ such that $v \geq 0$ define the function

$$\psi_v^{x_{ij}} : A_{\mathbf{k}}^{sp}(\overline{n}, r) \to A_{\mathbf{k}}^{sp}(\overline{n}, r)$$

to be the linear extension of

$$\psi_v^{x_{ij}}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\alpha,\gamma} \quad \alpha, \beta \in I(\overline{n}, r)$$

summed over all $\gamma \in I(\overline{n}, r)$ obtained from β by replacing some entries $\overline{j}$ by i and some entries $\overline{i}$ by j, such that the total number of replacements is v.

For $i, j \in I$ with $i < j$ and all $v, w \in \mathbf{Z}$ such that $v, w \geq 0$ define the symplectic function

$$\psi_{v,w}^{y_{ij}} : A_{\mathbf{k}}^{sp}(\overline{n}, r) \to A_{\mathbf{k}}^{sp}(\overline{n}, r)$$

to be the linear extension of

$$\psi_{v,w}^{y_{ij}}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\alpha,\gamma} \quad \alpha, \beta \in I(\overline{n}, r)$$

summed over all $\gamma \in I(\overline{n}, r)$ obtained from β by replacing v of the entries j by i and w of the entries $\overline{i}$ by $\overline{j}$.

For $i \in I$ and all $v \in \mathbf{Z}$ such that $v \geq 0$ define the symplectic function

$$\psi_v^{z_i} : A_{\mathbf{k}}^{sp}(\overline{n}, r) \to A_{\mathbf{k}}^{sp}(\overline{n}, r)$$

to be the linear extension of

$$\psi_v^{z_i}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\alpha,\gamma} \quad \alpha, \beta \in I(\overline{n}, r)$$

summed over all $\gamma \in I(\overline{n}, r)$ obtained from β by replacing v of the entries $\overline{i}$ by i.

For all three preceding definitions we take the empty sum to be zero.

3.7.3 Lemma. *Let $i, j \in I$ with $i < j$, let $t \in \mathbf{k}$ and let $\alpha, \beta \in I(\overline{n}, r)$. Then*

$$x_{i,j}(t) \circ d_{\alpha,\beta} = \sum_{v=0}^{r} t^v \psi_v^{x_{i,j}}(d_{\alpha,\beta}).$$

Proof. Let $i, j \in I$ with $i < j$, let $t \in \mathbf{k}$ and let $\alpha, \beta \in I(\overline{n}, r)$. For any $g \in Sp_{2n}(\mathbf{k})$

$$\begin{aligned}(x_{i,j}(t) \circ d_{\alpha,\beta})(g) &= d_{\alpha,\beta}(g x_{i,j}(t)) \\ &= \sum_{\gamma \in I(\overline{n}, r)} d_{\alpha,\gamma}(g) d_{\gamma,\beta}(x_{i,j}(t))\end{aligned}$$

by the rules of matrix multiplication. So

$$x_{i,j}(t) \circ d_{\alpha,\beta} = \sum_{\gamma \in I(\overline{n},r)} d_{\gamma,\beta}(x_{i,j}(t)) d_{\alpha,\gamma}.$$

However $d_{\gamma,\beta}(x_{i,j}(t)) = 0$ unless

$$(\gamma_\rho, \beta_\rho) \in \{(1,1), \ldots, (n,n), (\overline{n},\overline{n}), \ldots, (\overline{1},\overline{1})(i,\overline{j}), (j,\overline{i})\}$$

for all $\rho \in \{1, \ldots, r\}$. In this case $d_{\gamma,\beta}(x_{i,j}(t)) = t^v$ where v is the number indices ρ such that $(\gamma_\rho, \beta_\rho) \in \{(i,\overline{j}), (j,\overline{i})\}$ and γ is obtained from β by replacing some entries $\overline{j}$ by i and some entries $\overline{i}$ by j with a total of v replacements. We must have $v \leq r$ since there are only r elements in β, and the result follows.

□

3.7.4 Lemma. *Let $i,j \in I$ with $i < j$, let $t \in \mathbf{k}$ and let $\alpha, \beta \in I(\overline{n}, r)$. Then*

$$y_{i,j}(t) \circ d_{\alpha,\beta} = \sum_{v=0}^{r} \sum_{w=0}^{r} (-1)^w t^{v+w} \psi_{v,w}^{y_{i,j}}(d_{\alpha,\beta}).$$

Proof. Let $i,j \in I$ with $i < j$, let $t \in \mathbf{k}$ and let $\alpha, \beta \in I(\overline{n}, r)$. Then

$$y_{i,j}(t) \circ d_{\alpha,\beta} = \sum_{\gamma \in I(\overline{n},r)} d_{\alpha,\gamma}(g) d_{\gamma,\beta}(y_{i,j}(t))$$

by the rules of matrix multiplication. However $d_{\gamma,\beta}(y_{i,j}(t)) = 0$ unless

$$(\gamma_\rho, \beta_\rho) \in \{(1,1), \ldots, (n,n), (\overline{n},\overline{n}), \ldots, (\overline{1},\overline{1})(i,j), (\overline{j},\overline{i})\}$$

for all $\rho \in \{1, \ldots, r\}$. In this case $d_{\gamma,\beta}(y_{i,j}(t)) = (-1)^w t^{v+w}$ where v is the number indices ρ such that $(\gamma_\rho, \beta_\rho) = (i,j)$ and w is the number of indices ρ such that $(\gamma_\rho, \beta_\rho) = (\overline{j},\overline{i})$. Since β contains r entries neither v nor w will exceed r. The result follows.

□

3.7.5 Lemma. *Let $i \in I$, let $t \in \mathbf{k}$ and let $\alpha, \beta \in I(\overline{n}, r)$. Then*

$$z_i(t) \circ d_{\alpha,\beta} = \sum_{v=0}^{r} t^v \psi_v^{z_i}(d_{\alpha,\beta}).$$

Proof. Let $i \in I$, let $t \in \mathbf{k}$ and let $\alpha, \beta \in I(\overline{n}, r)$. Then

$$z_i(t) \circ d_{\alpha,\beta} = \sum_{\gamma \in I(\overline{n},r)} d_{\alpha,\gamma}(g) d_{\gamma,\beta}(z_i(t))$$

by the rules of matrix multiplication. However $d_{\gamma,\beta}(z_i(t)) = 0$ unless

$$(\gamma_\rho, \beta_\rho) \in \{(1,1), \ldots, (n,n), (\overline{n}, \overline{n}), \ldots, (\overline{1}, \overline{1})(i, \overline{i})\}$$

for all $\rho \in \{1, \ldots, r\}$. In this case $d_{\gamma,\beta}(z_i(t)) = t^v$ where v is the number indices ρ such that $(\gamma_\rho, \beta_\rho) = (i, \overline{i})$. As β contains r elements $v \leq r$, and the result follows.

□

4

Tableaux and Dimension Formulae.

In the case of $GL_n(\mathbf{C})$ the dimension of the irreducible module $V_{\lambda,\mathbf{C}}$ of highest weight λ has been shown to be equal to the number of semistandard λ-tableaux with entries from $\{1,\ldots,n\}$. Moreover a basis can be constructed which is indexed by these tableaux in a natural way. We wish to do the same for the symplectic group, using the definition of symplectic tableaux given by King and El-Sharkaway in [1].

Let λ be an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

4.1 Symplectic Tableaux.

Let $I = \{1,\ldots,n\}$ and $\bar{I} = \{\bar{1},\ldots,\bar{n}\}$. Define an ordering on $I \cup \bar{I}$ by

$$1 < \bar{1} < 2 < \bar{2} < \ldots < n < \bar{n}.$$

4.1.1 Definition. A *symplectic* λ-tableau is a λ-tableau with entries from $I \cup \bar{I}$ which satisfies the following three conditions:-

(i) the entries are non-decreasing from left to right along the rows;
(ii) the entries are strictly increasing from top to bottom down the columns;
(iii) the entries of the i^{th} row are greater or equal to i for all i.

Example. Consider $Sp_4(\mathbf{C})$ with $\lambda = (3,2)$. The $(3,2)$-tableau $\begin{array}{|c|c|c|}\hline 1 & \bar{1} & 2 \\ \hline \bar{1} & 2 \\ \cline{1-2}\end{array}$ is semistandard but not symplectic since there is an entry $\bar{1}$ in the second row. However, both $\begin{array}{|c|c|c|}\hline 1 & \bar{1} & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & \bar{2} \\ \hline 2 & \bar{2} \\ \cline{1-2}\end{array}$ are symplectic.

4.2 A Dimension Formula.

Let V_λ^{Sp} be an irreducible $Sp_{2n}(\mathbf{C})$-module of highest weight λ. In King and El-Sharkaway [1] p 3156-3157, the authors describe branching rules corresponding to group-subgroup restrictions. They show that

$$V_\lambda^{Sp_{2n}(\mathbf{C})}|_{Sp_{2n-2}(\mathbf{C})} = V_{\mu_1}^{Sp_{2n-2}(\mathbf{C})} \oplus \ldots \oplus V_{\mu_k}^{Sp_{2n-2}(\mathbf{C})}$$

where $\mu_1, \ldots, \mu_k$ are the partitions whose diagrams are obtained by twice removing squares from the λ-diagram in all possible admissible ways, such that at most $n-1$ rows remain. A way of removing squares is admissible if no more than one square is removed from a single column, and if the remaining diagram is of a dominant partition. Let μ be a partition whose diagram can be obtained in this way. Clearly, there may be more than one way to remove squares twice, in admissible ways, and obtain the diagram of μ. The number of ways of doing so is the multiplicity of the term $V_\mu^{Sp_{2n-2}(\mathbb{C})}$ in the decomposition.

4.2.1 Theorem. *The dimension of* $V_\lambda^{Sp_{2n}(\mathbb{C})}$ *is equal to the number of symplectic* λ-*tableaux with entries from* $\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$.

Proof. This is proved in King and El-Sharkaway [1], and we give an indication of the proof here. Note that we have interchanged the use of i and $\overline{i}$ by King.

Let $V_{\mu_i}^{Sp_{2n-2}(\mathbb{C})}$ be one of the terms on the right hand side of

$$V_\lambda^{Sp_{2n}(\mathbb{C})}|_{Sp_{2n-2}(\mathbb{C})} = V_{\mu_1}^{Sp_{2n-2}(\mathbb{C})} \oplus \ldots \oplus V_{\mu_k}^{Sp_{2n-2}(\mathbb{C})}.$$

Then the μ_i-diagram can be obtained from the λ-diagram by removing squares in admissible ways in two stages, with at most $n-1$ rows left after both have been completed. The particular way this was done determines the i^{th} term $V_{\mu_i}^{Sp_{2n-2}(\mathbb{C})}$ uniquely. Let T be a λ-diagram. As we produce the μ_i-diagram in this particular way, put $\overline{n}$ in every square in T removed from the λ-diagram in the first stage, and n in every square in T removed from the λ-diagram in the second stage. Then T determines the i^{th} term uniquely.

Now consider the decomposition

$$V_{\mu_i}^{Sp_{2n-2}(\mathbb{C})}|_{Sp_{2n-4}(\mathbb{C})} = V_{\nu_1}^{Sp_{2n-4}(\mathbb{C})} \oplus \ldots \oplus V_{\nu_l}^{Sp_{2n-4}(\mathbb{C})}.$$

Then $\nu_1, \ldots, \nu_l$ are the partitions of the diagrams obtained from the μ_i-diagram by twice removing squares in all admissible ways, such that at most $n-2$ rows remain. A particular term $V_{\nu_j}^{Sp_{2n-4}(\mathbb{C})}$ corresponds to a particular way of doing this. Put an entry $\overline{n-1}$ in the squares in T that were removed from the μ_i-diagram in stage 1 of producing the ν_j-diagram, and an entry $n-1$ in the squares in T that were removed from the μ_i-diagram in the second stage. Then T determines the term $V_{\nu_j}^{Sp_{2n-4}(\mathbb{C})}$ uniquely.

We can now go on to decompose $V_{\nu_j}^{Sp_{2n-4}(\mathbb{C})}$ into a direct sum of irreducible $Sp_{2n-6}(\mathbb{C})$-modules, and so on. Continuing in this way we eventually reach a term in a decomposition which has been obtained by removing all the squares in a diagram. This is called a final term, and corresponds to an irreducible module for a maximal torus $H \subset Sp_{2n}(\mathbb{C})$. It is therefore one-dimensional. Hence the total number of all final terms is the dimension of $V_\lambda^{Sp_{2n}(\mathbb{C})}$.

Choose a particular final term. If we have put entries in T at each stage of the process in which the final term was obtained by removing squares of the λ-diagram, then every square in T has an entry from $I \cup \overline{I}$. We claim that T is symplectic.

A way of removing squares from a diagram is admissible only if the diagram remaining is that of a dominant partition. Therefore, no square can be removed at an earlier stage than those to its right in the same row. So the entries in the same row to the right of any square are not less than the entry in that square. Hence, the entries of T are non-decreasing from left to right along rows.

A way of removing squares is not admissible if more than one square is removed from the same column. Since the remaining diagram must also be of a dominant partition, if a square is removed from a column it must be the bottom square. Hence the entry in a square is always strictly greater than the entry in the square directly above it. So the entries within the columns of T are strictly increasing from top to bottom.

Let $i \in \{1, \ldots, n\}$. After the stages when i and $\bar{i}$ are put into the squares of T, the diagram produced by removing from the λ-diagram all the filled squares in T must contain at most $i-1$ rows. Therefore, the entries of the squares in row i of T are all greater than or equal to i. Therefore, T is a symplectic λ-tableau.

Conversely, let T be any symplectic λ-tableau with entries from $I \cup \bar{I}$. Then the entries of T give a recipe for removing all the squares in the λ-diagram such that at each stage the way of removing them is admissible, and such that the partition diagrams left at every second stage have no more than the allowed number of rows. Hence T corresponds uniquely to a final term in the decomposition of $V_\lambda^{Sp_{2n}(\mathbb{C})}$ as a direct sum of irreducible H-modules.

So there is a one-to-one correspondence between the one-dimensional weight spaces of $V_\lambda^{Sp_{2n}(\mathbb{C})}$ with multiplicities, and the set of symplectic λ-tableaux with entries from $I \cup \bar{I}$. $\square$

Example. Let $n = 2$ and let $\lambda = (2,1)$. The dimension of the irreducible rational module for $Sp_4(\mathbb{C})$ of highest weight λ is equal to the number of symplectic $(2,1)$-tableaux with entries from $\{1, \bar{1}, 2, \bar{2}\}$. These tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{2} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline \bar{2} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{2} \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline \bar{1} & \bar{1} \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline \bar{1} & \bar{1} \\ \hline \bar{2} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline \bar{1} & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline \bar{1} & 2 \\ \hline \bar{2} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline \bar{1} & \bar{2} \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline \bar{1} & \bar{2} \\ \hline \bar{2} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{2} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline \bar{2} \\ \cline{1-1}\end{array}$$

and so the dimension is 16.

4.3 Another Dimension Formula.

Recall from Chapter 1 that Weyl gives a formula for the dimension of the irreducible rational module of highest weight λ of a simple algebraic group over $\mathbb{C}$.

Suppose F_λ is an irreducible rational $Sp_{2n}(\mathbb{C})$-module of highest weight $\lambda \in X(T)^+$. Let Φ denote the set of roots for $Sp_{2n}(\mathbb{C})$, and let Φ^+ denote the set of positive roots. For any $\alpha \in \Phi$, let α^ν denote the coroot corresponding to α. Let $\rho \in X(T)$ be given by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi+} \alpha.$$

Then the dimension of F_λ is given by

$$\dim F_\lambda = \prod_{\alpha \in \Phi+} \frac{(\lambda + \rho, \alpha^\nu)}{(\rho, \alpha^\nu)}.$$

Example. Consider the group $Sp_4(\mathbb{C})$. Let $T \subset Sp_4(\mathbb{C})$ be the subset of diagonal matrices. Then an element $t \in T$ is of the form

$$t = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & t_2^{-1} & \\ 0 & & & t_1^{-1} \end{pmatrix} \qquad t_1, t_2 \in \mathbb{C} \setminus \{0\}.$$

The set of weights $X(T)$ is generated by two elements e_1 and e_2 where $e_i(t) = t_i$ for $i \in \{1,2\}$. A set of simple roots for $Sp_4(\mathbb{C})$ is $\Pi = \{\alpha_1, \alpha_2\}$ where

$$\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= 2e_2. \end{aligned}$$

The corresponding set of positive roots Φ^+ is given by

$$\Phi^+ = \{\ \alpha_1\ ,\ \alpha_2\ ,\ \alpha_1 + \alpha_2\ ,\ 2\alpha_1 + \alpha_2\ \}$$

and so $\rho = \frac{1}{2}(4e_1 + 2e_2) = 2e_1 + e_2$.

Let $\lambda = (2,1)$. Then $\lambda = 2e_1 + e_2 = \rho$, and

$$\dim F_{(2,1)} = \prod_{\alpha \in \Phi+} \frac{(2\rho, \alpha^\nu)}{(\rho, \alpha^\nu)} = \prod_{\alpha \in \Phi+} 2 = 2^4 = 16.$$

We describe another formula for the dimension of F_λ, derived from Weyl's dimension formula in El-Samra and King [1].

4.3.1 Definition. Corresponding to each square in the λ-diagram we define an integer called the *hook length*, and it is the sum of the number of squares to the right in the same row and the number of squares below in the same column and 1 (for the square itself).

For example, if we write the value of the hook length in each square of the diagram, then when $\lambda = (2,1)$ these are

$$\begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Let $H(\lambda)$ denote the product of all the hook lengths in the λ-diagram. Then $H((2,1)) = 3$. Let μ denote the conjugate partition to λ. Then each pair i, j with $i \in \{1, \ldots, \mu_1\}$ and $j \in \{1, \ldots, \lambda_i\}$ determines a square in the λ-diagram, that is, the square in the i^{th} row and j^{th} column.

The formula given by El-Samra and King for the dimension of the irreducible rational $Sp_{2n}(\mathbb{C})$-module F_λ of highest weight λ is the following.

$$\dim F_\lambda = \frac{\prod_{i>j}^{\lambda}(2n + \lambda_i + \lambda_j - i - j + 2)\prod_{i\leq j}^{\lambda}(2n - \mu_i - \mu_j + i + j)}{H(\lambda)},$$

with one factor in the numerator for each square in the λ-diagram.

Example. Again let $n = 2$ and $\lambda = (2, 1)$, and so $\mu = (2, 1)$. If we write each factor in the above formula inside its corresponding square we have

$$\dim F_{(2,1)} = \frac{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} = \frac{2 \times 4 \times 6}{3 \times 1 \times 1} = \frac{48}{3} = 16.$$

5

Weyl Modules.

In this chapter we study a module for $Sp_{2n}(\mathbf{k})$ which is a Weyl module in the sense of Section 1.12. To do this we regard $Sp_{2n}(\mathbf{k})$ as a subgroup of $GL_{2n}(\mathbf{k})$, and find an $Sp_{2n}(\mathbf{k})$-submodule of the Weyl module $V_{\lambda,\mathbf{k}}$ described in Chapter 2. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a weight of $Sp_{2n}(\mathbf{k})$ then it can be identified with the weight $\lambda = (\lambda_1, \ldots, \lambda_{2n})$ of $GL_{2n}(\mathbf{k})$, where $\lambda_{n+1} = \ldots = \lambda_{2n} = 0$. If λ is a dominant weight of $Sp_{2n}(\mathbf{k})$ then it is a dominant weight of $GL_{2n}(\mathbf{k})$.

Throughout this chapter let λ be an integer partition of the form $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 + \ldots + \lambda_n = r$ and $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, and let $\mathbf{k}$ be any infinite field.

5.1 The Action of the Hyperalgebra.

Let V' be a vector space of dimension $2n$ over $\mathbb{C}$. There is a natural action of $GL_{2n}(\mathbb{C})$ on V' and therefore also of $Sp_{2n}(\mathbb{C})$. Let $T(V') = \bigoplus_{i \geq 0} T^i(V')$ where

$$T^i(V') = \underbrace{V' \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} V'}_{i \text{ times}}$$

as before. Then $GL_{2n}(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ act on $T(V')$, and on $T^r(V')$, by acting on each component of a tensor separately. The group S_r of permutations on r elements acts on $T^r(V')$ by

$$\sigma(v_1 \otimes \ldots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(r)}$$

for all $\sigma \in S_r$ and all $v_1, \ldots, v_r \in V'$. Let $\{v_1, \ldots, v_n, v_{\overline{n}}, \ldots, v_{\overline{1}}\}$ be a symplectic basis of V', considered as an $Sp_{2n}(\mathbb{C})$-module. Let $I = \{1, \ldots, n\}$ and $\overline{I} = \{\overline{1}, \ldots, \overline{n}\}$, and for all $i, j \in I \cup \overline{I}$ let $E_{i,j}$ denote the matrix with $(i,j)^{\text{th}}$ coefficient 1 and all other coefficients 0. The Lie algebra $\mathfrak{gl}_{2n}(\mathbb{C}) = \text{Lie}(GL_{2n}(\mathbb{C}))$ inherits an action on V' in which $E_{i,j} \in \mathfrak{gl}_{2n}(\mathbb{C})$ acts as the unique derivation satisfying

$$E_{i,j} v_k = \begin{cases} v_i & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in I \cup \overline{I}$. The Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C}) = \text{Lie}(Sp_{2n}(\mathbb{C}))$ can be embedded in $\mathfrak{gl}_{2n}(\mathbb{C})$ and has the same action on V'. In this way $T(V')$ and $T^r(V')$ are modules for $\mathfrak{gl}_{2n}(\mathbb{C})$ and $\mathfrak{sp}_{2n}(\mathbb{C})$.

The action of $\mathfrak{gl}_{2n}(\mathbb{C})$ on $T^r(V')$ extends canonically to an action of its universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_{2n}(\mathbb{C}))$ on $T^r(V')$. In this way, the Kostant $\mathbb{Z}$-forms $\mathcal{U}_{\mathbb{Z}}^{gl}$ and $\mathcal{U}_{\mathbb{Z}}^{sp}$ of the universal enveloping algebras of $\mathfrak{gl}_{2n}(\mathbb{C})$ and $\mathfrak{sl}_{2n}(\mathbb{C})$ respectively have a natural action on $T^r(V')$. Let $e_{i,j}$ denote the element corresponding to $E_{i,j}$ under the natural embedding of $\mathfrak{gl}_{2n}(\mathbb{C})$ in $\mathcal{U}(\mathfrak{gl}_{2n}(\mathbb{C}))$. Then $\mathcal{U}_{\mathbb{Z}}^{gl}$ has a $\mathbb{Z}$-basis consisting of elements of the form

$$\left(\prod_{(i,j)\in\mathcal{I}} \frac{e_{j,i}^{p_{i,j}}}{p_{i,j}!}\right)\left(\prod_{i\in I\cup\overline{I}} \binom{e_{i,i}}{s_i}\right)\left(\prod_{(i,j)\in\mathcal{I}} \frac{e_{i,j}^{q_{i,j}}}{q_{i,j}!}\right),$$

where the $p_{i,j}, q_{i,j}$ and s_i are all non-negative integers, where the set $\mathcal{I}$ consists of all pairs $(i,j) \in I \cup \overline{I}$ such that $i < j$, according to the ordering $1 < \ldots < n < \overline{n} < \ldots < \overline{1}$, and where the products are taken in a fixed order. The element $e_{i,j}$ acts in the same way as $E_{i,j}$, that is, as the unique derivation satisfying

$$e_{i,j}v_k = \begin{cases} v_i & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in I \cup \overline{I}$.

Let $I(\overline{n}, r)$ denote the set of r-tuples with entries from $I \cup \overline{I}$. For any $\alpha \in I(\overline{n}, r)$ of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ let $v_\alpha \in T^r(V')$ be the tensor given by

$$v_\alpha = v_{\alpha_1} \otimes v_{\alpha_2} \otimes \ldots \otimes v_{\alpha_r}.$$

Then $I(\overline{n}, r)$ forms an indexing set for the basis elements of $T^r(V')$.

Let $i, j \in I \cup \overline{I}$ with $i \neq j$, and let $\alpha \in I(\overline{n}, r)$. Then $v_\alpha \in T^r(V')$ and

$$e_{i,j}v_\alpha = \sum_\beta v_\beta$$

where the sum is over all $\beta \in I(\overline{n}, r)$ obtained from α by replacing one entry j by i. If no such β exist then $e_{i,j}v_\alpha = 0$. Let $m \in \mathbb{N}$ then

$$(e_{i,j})^m v_\alpha = m! \sum_\beta v_\beta$$

where the sum is over all $\beta \in I(\overline{n}, r)$ obtained from α by replacing m entries j by i. Given such a β there are $m!$ different orders in which the entries of α could be changed to give β, and so we have the multiplier $m!$. If no such β exists then $(e_{i,j})^m v_\alpha = 0$.

Let $i \in I \cup \overline{I}$, and let $\alpha \in I(\overline{n}, r)$. Let N be the number of entries i in α. Then

$$e_{i,i}v_\alpha = Nv_\alpha.$$

Let $m \in \mathbb{N}$. Then

$$\begin{aligned}\binom{e_{i,i}}{m} v_\alpha &= \frac{(e_{i,i})(e_{i,i}-1)\ldots(e_{i,i}-m+1)}{m!} v_\alpha \\ &= \frac{N(N-1)\ldots(N-m+1)}{m!} v_\alpha\end{aligned}$$

If $N < m$ then one of the factors in the numerator is zero. So

$$\binom{e_{i,i}}{m} v_\alpha = \begin{cases} \binom{N}{m} v_\alpha & N \geq m \\ 0 & N < m \end{cases}$$

5.1.1 Definition. Let V denote the free $\mathbf{Z}$-module of rank $2n$ generated by the vectors $v_1, \ldots, v_n, v_{\overline{n}}, \ldots, v_{\overline{1}}$.

Then V forms a lattice in V', and by the above calculations we see that V is stable under the action of $\mathcal{U}_{\mathbf{Z}}^{gl}$ and therefore also under the action of $\mathcal{U}_{\mathbf{Z}}^{sp}$.

Recall from Chapter 2 the element $\phi_\lambda \in T^r(V)$ defined by

$$\phi_\lambda = \alpha t_{T_0} = \begin{vmatrix} v_1 & v_1 & v_1 & \ldots & v_1 \\ v_2 & v_2 & v_2 & \ldots & v_2 \\ v_3 & v_3 & v_3 & \ldots & v_3 \\ \vdots & \vdots & \vdots & & \vdots \\ v_{\mu_1} & v_{\mu_1} & v_{\mu_1} & \cdots & v_{\mu_1} \end{vmatrix} \otimes \begin{vmatrix} v_1 & v_1 & \ldots & v_1 \\ v_2 & v_2 & \ldots & v_2 \\ \vdots & \vdots & & \vdots \\ v_{\mu_2} & v_{\mu_2} & \cdots & v_{\mu_2} \end{vmatrix} \otimes \ldots \otimes \begin{vmatrix} v_1 & \ldots & v_1 \\ \vdots & & \vdots \\ v_{\mu_{\lambda_1}} & \cdots & v_{\mu_{\lambda_1}} \end{vmatrix}$$

where $\alpha = \sum_{\sigma \in C(\lambda)} \mathrm{sign}(\sigma)\sigma$ and T_0 is the basic λ-tableau.

5.1.2 Definitions. Let $V_{\lambda,\mathbf{Z}}^{gl} \in T^r(V)$ be the $\mathcal{U}_{\mathbf{Z}}^{gl}$-module generated by ϕ_λ. That is,

$$V_{\lambda,\mathbf{Z}}^{gl} = \mathcal{U}_{\mathbf{Z}}^{gl} \phi_\lambda.$$

We define a $\mathcal{U}_{\mathbf{Z}}^{sp}$-submodule $V_{\lambda,\mathbf{Z}}^{sp} \subset V_{\lambda,\mathbf{Z}}^{gl}$ by

$$V_{\lambda,\mathbf{Z}}^{sp} = \mathcal{U}_{\mathbf{Z}}^{sp} \phi_\lambda.$$

5.1.3 Definition. Let $\mathbf{k}$ be any infinite field. Let $\overline{V}$ be the vector space over $\mathbf{k}$ of dimension $2n$ given by

$$\overline{V} = \mathbf{k} \otimes_{\mathbf{Z}} V.$$

Let $\mathcal{U}_{\mathbf{k}}^{gl} = \mathbf{k} \otimes_{\mathbf{Z}} \mathcal{U}_{\mathbf{Z}}^{gl}$ denote the hyperalgebra of $\mathfrak{gl}_{2n}(\mathbf{k})$, and let $\mathcal{U}_{\mathbf{k}}^{sp} = \mathbf{k} \otimes_{\mathbf{Z}} \mathcal{U}_{\mathbf{Z}}^{sp}$ denote the hyperalgebra of $\mathfrak{sp}_{2n}(\mathbf{k})$. Then $\mathcal{U}_{\mathbf{k}}^{\mathfrak{gl}}$ and $\mathcal{U}_{\mathbf{k}}^{\mathfrak{sp}}$ act on $\overline{V}$ and $T^r(\overline{V})$ in a natural way.

There is a natural map from $T^r(V)$ to $T^r(\overline{V})$ which sends an element $v \in T^r(V)$ to $1 \otimes v \in T^r(\overline{V})$, and we denote the image of ϕ_λ under this map by $\overline{\phi_\lambda}$. In general we will write a bar over an object in V or $T^r(V)$ to indicate the corresponding object in $\overline{V}$ or $T^r(\overline{V})$. For instance, $\{\overline{v_1}, \ldots, \overline{v_n}, \overline{v_{\overline{n}}}, \ldots, \overline{v_{\overline{1}}}\}$ forms a basis for $\overline{V}$, where $\overline{v_i} = 1 \otimes v_i$ for $i \in I \cup \overline{I}$.

Recall Carter and Lusztig's module $V_{\lambda,\mathbf{k}}$, described in Chapter 2, which satisfies

$$V_{\lambda,\mathbf{k}} = \mathcal{U}_{\mathbf{k}}^{gl}\overline{\phi_\lambda}.$$

Henceforth we write $V_{\lambda,\mathbf{k}}^{gl}$ instead of $V_{\lambda,\mathbf{k}}$ for clarity.

5.1.4 Definition. Let the submodule $V_{\lambda,\mathbf{k}}^{sp} \subset V_{\lambda,\mathbf{k}}^{gl}$ be given by

$$V_{\lambda,\mathbf{k}}^{sp} = \mathcal{U}_{\mathbf{k}}^{sp}\overline{\phi_\lambda}.$$

We call this the *symplectic Weyl module.*

Since $\mathcal{U}_{\mathbf{k}}^{sp}$-modules are also $Sp_{2n}(\mathbf{k})$-modules and *vice versa,* $V_{\lambda,\mathbf{k}}^{sp}$ is a cyclic $Sp_{2n}(\mathbf{k})$-module generated by $\overline{\phi_\lambda}$. It is also a polynomial module since it is a subspace of $T^r(\overline{V})$.

5.1.5 Lemma. *The tensor $\overline{\phi_\lambda} \in T^r(\overline{V})$ has weight λ with respect to the subgroup of diagonal matrices $S \subset GL_{2n}(\mathbf{k})$, and with respect to the subgroup of diagonal matrices $T \subset Sp_{2n}(\mathbf{k})$.*

Proof. Let $s \in S$. Then S is of the form

$$\begin{pmatrix} s_1 & & & & & \\ & \ddots & & & 0 & \\ & & s_n & & & \\ & & & s_{\overline{n}} & & \\ & 0 & & & \ddots & \\ & & & & & s_{\overline{1}} \end{pmatrix}$$

where $s_1, \ldots, s_n, s_{\overline{n}}, \ldots, s_{\overline{1}} \in \mathbf{k} \setminus \{0\}$. Then for all $i \in I \cup \overline{I}$ $s\overline{v_i} = s_i\overline{v_i}$, and $\overline{v_i}$ occurs in each tensor in $\overline{\phi_\lambda}$ exactly λ_i times. So $s\overline{\phi_\lambda} = s_1^{\lambda_1} \ldots s_n^{\lambda_n}\overline{\phi_\lambda} = \lambda(s)\overline{\phi_\lambda}$.

The subgroup $T \subset Sp_{2n}(\mathbf{k})$ of diagonal matrices consists of elements $t \in T$ of the form

$$\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & 0 & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix}$$

where $t_1, \ldots, t_n \in \mathbf{k} \setminus \{0\}$. Since λ has at most n parts, $\overline{\phi_\lambda}$ has weight λ with respect to the action of $T \subset Sp_{2n}(\mathbf{k})$.

$\square$

5.2 A Linearly Independent Subset.

In this section we shall construct an element $v_T \in V^{sp}_{\lambda,\mathbb{Z}}$ for every symplectic λ-tableau T. Henceforth all λ-tableaux are assumed to contain entries from $I \cup \overline{I}$. We begin by considering the action of $\mathcal{U}^{sp}_{\mathbb{Z}}$ on ϕ_λ.

Let $i, j \in I \cup \overline{I}$, and let $\alpha \in I(\overline{n}, r)$ be of the form $\alpha = (\alpha_1, \ldots, \alpha_r)$. Then the action of $e_{i,j} \in \mathcal{U}^{gl}_{\mathbb{Z}}$ on $T^r(V)$ is given by

$$e_{i,j}(v_{\alpha_1} \otimes \ldots \otimes v_{\alpha_r}) = \sum_\beta v_{\beta_1} \otimes \ldots \otimes v_{\beta_r}$$

where the sum is over all $\beta \in I(\overline{n}, r)$ obtained from α by replacing one entry j by i. For example,

$$e_{1,2}(v_1 \otimes v_2 \otimes v_{\overline{3}} \otimes v_2) = v_1 \otimes v_1 \otimes v_{\overline{3}} \otimes v_2 + v_1 \otimes v_2 \otimes v_{\overline{3}} \otimes v_1.$$

Let T be any λ-tableau. Let $t_T \in T^r(V)$ denote the tensor $v_{t_1} \otimes v_{t_2} \otimes \ldots \otimes v_{t_r}$ where t_i is the entry in the i^{th} position of T. Then $\phi_T = \alpha t_T$ is a sum of tensors which are all permutations of t_T. Hence for any $i, j \in I \cup \overline{I}$

$$e_{i,j}\phi_T = \sum_{T'} \phi_{T'}$$

summed over all λ-tableaux T' which are obtained from T by replacing one entry j by i.

For example, when $n = 3$ and $\lambda = (2, 2, 1)$ then

$$e_{\overline{3},2}\phi_\lambda = e_{\overline{3},2}\phi \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$= \phi \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \overline{3} \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} + \phi \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \overline{3} & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

5.2.1 Lemma. *Let T be a λ-tableau and let $i, j \in I \cup \overline{I}$. For any $m \in \mathbb{N}$*

$$e^m_{i,j}\phi_T = m! \sum_{T'} \phi_{T'}$$

summed over all λ-tableaux T' which are obtained from T by replacing m entries j by i.

Proof. We have shown this is true for $m = 1$. For some $k \in \mathbb{N}$, such that T contains more than k entries equal to j, assume that

$$e^k_{i,j}\phi_T = k! \sum_{T_k} \phi_{T_k}$$

summed over all $T_k \in \mathcal{T}$ obtained from T by replacing k entries j by i. Then

$$e_{i,j}(e_{i,j}^k \phi_T) = k! \sum_{T_k} \sum_{T'} \phi_{T'}$$

where the inner sum is over all T' obtained from T_k by replacing one entry j by i.

Let T_{k+1} be a λ-tableau obtained from T by replacing $k+1$ entries j by i. If we change one of those $k+1$ entries back to a j we obtain one of the λ-tableaux T_k. There are $k+1$ ways of doing this, and so T_{k+1} appears as T' for $k+1$ of the λ-tableaux T_k in the sum. So

$$\begin{aligned} e_{i,j}^{k+1} \phi_T &= k!\,(k+1) \sum_{T_{k+1}} \phi_{T_{k+1}} \\ &= (k+1)! \sum_{T_{k+1}} \phi_{T_{k+1}} \end{aligned}$$

summed over all T_{k+1} obtained from T by replacing $k+1$ entries j by i. So by induction the result is true for all $m \in \mathbb{N}$ such that T contains at least m entries equal to j.

However, if m is greater than the number of entries equal to j in T then $e_{i,j}^m \phi_T = 0$ because there are no tableaux obtained from T by replacing m entries of j by i. If we set the empty sum equal to zero then the result holds.

□

5.2.2 Definition. Let $\mathcal{T}$ denote the set of λ-tableaux with entries from $I \cup \bar{I}$.

5.2.3 Lemma. *Let $i, j \in I \cup \bar{I}$, $T \in \mathcal{T}$ and $m \in \mathbb{N}$ satisfy:-*

(i) both i and j do not occur within the same row in T;

(ii) T contains at least m entries equal to j.

Then

$$e_{i,j}^m \psi_T = m! \sum_{T'} \psi_{T'}$$

where the sum is over all λ-tableaux T' such that T' is non-decreasing along rows and is a row permutation of a λ-tableau obtained from T by replacing m entries j by i.

Proof.

$$\psi_T = \sum_{\sigma \in S} \sigma(\phi_T) = \sum_{\sigma \in S} \phi_{\sigma(T)}$$

where $S \subset S_r$ is the subgroup of row permutations for which $\sigma(T) \neq T$. Then

$$e_{i,j}^m \psi_T = \sum_{\sigma \in S} e_{i,j}^m \phi_{\sigma(T)}.$$

Since the action of S_r commutes with the action of $\mathcal{U}_{\mathbf{Z}}^{sp}$ this is equal to

$$\sum_{\sigma\in S}\sigma(e_{i,j}^{m}\phi_{(T)}) = m!\sum_{\sigma\in S}\sigma\sum_{T'}\phi_{T'}$$

where the inner sum is over all T' obtained from T by replacing m entries j by i. The set of such T' can be partitioned into subsets of tableaux in which the same number of replacements have been made in each row.

For a given number of replacements j by i in each row, with a total of m replacements, let $T_1',\ldots,T_t'$ be all the λ-tableaux obtained from T in this way. We wish to show

$$\sum_{\sigma\in S}\sum_{k=1}^{t}\phi_{\sigma(T_k')} = \psi_{T_1'}.$$

Let $S'\subset S_r$ be the subgroup of row permutations σ of T_1' such that $\sigma(T_1')\neq T_1'$. Then

$$\psi_{T_1'} = \sum_{\sigma'\in S'}\phi_{\sigma'(T_1')}.$$

Now T_1' is identical to T in the squares which do not contain entries j or i, and since T doesn't contain both i and j within any row $S\subset S'$. Also $T_1',\ldots,T_t'$ are all the row permutations of T_1' which fix entries other than i and j. Hence the set of permutations $\{\sigma_1,\ldots,\sigma_t\}$ satisfying

$$T_1' = \sigma_1(T_1'),\ T_2' = \sigma_2(T_1'),\ \ldots,\ T_t' = \sigma_t(T_1')$$

forms a right transversal of S in S'. Therefore

$$\begin{aligned}\sum_{\sigma\in S}\sum_{k=1}^{t}\phi_{\sigma(T_k')} &= \sum_{\sigma\in S}\sum_{k=1}^{t}\phi_{\sigma\sigma_k(T_1')}\\ &= \sum_{\sigma'\in S'}\phi_{\sigma'(T_1')}\\ &= \psi_{T_1'}.\end{aligned}$$

Let T^* be the unique λ-tableau which is a row permutation of T_1' and is non-decreasing along its rows. Then

$$\psi_{T_1'} = \psi_{T^*}$$

and the lemma is proved.

□

Recall the decomposition

$$\mathcal{U}_{\mathbf{Z}}^{sp} = \mathcal{U}_{sp,\mathbf{Z}}^{-}\mathcal{U}_{sp,\mathbf{Z}}^{0}\mathcal{U}_{sp,\mathbf{Z}}^{+}.$$

5.2.4 Lemma. *Any non-trivial basis element $b \in \mathcal{U}^+_{sp,\mathbf{Z}}$ satisfies*

$$b.\phi_\lambda = 0.$$

Proof. $\mathcal{U}^+_{sp,\mathbf{Z}}$ is $\mathbf{Z}$-spanned by the elements

$$\left(\prod_{i\in I}\frac{z_i^{\zeta_i}}{\zeta_i!}\right)\left(\prod_{\substack{i,j\in I\\ i<j}}\frac{x_{i,j}^{\chi_{i,j}}}{\chi_{i,j}!}\frac{y_{i,j}^{\varphi_{i,j}}}{\varphi_{i,j}!}\right)$$

where ζ_i, $\chi_{i,j}$ and $\varphi_{i,j}$ are non-negative integers. Let $i,j \in I$ with $i<j$ and let $\varphi \in \mathbf{N}$. Then

$$\begin{aligned}\frac{y_{i,j}^{\varphi}}{\varphi!}\phi_\lambda &= \frac{(e_{i,j}-e_{\bar{j},\bar{i}})^{\varphi}}{\varphi!}\phi_{T_0}\\ &= \frac{e_{i,j}^{\varphi}}{\varphi!}\phi_{T_0}\end{aligned}$$

since T_0 contains no entries equal to $\bar{i}$. This equals

$$\sum_{T'}\phi_{T'}$$

summed over all T' obtained from T by replacing φ entries j by i. If $j > \mu_1$then there are less than j rows in T_0. T_0 has all entries equal to k in the k^{th} row for all k, and so T_0 contains no entries equal to j. Hence $\frac{1}{\varphi!}e_{i,j}^{\varphi}\phi_{T_0} = 0$. So assume $j \le \mu_1$. If $\varphi > \lambda_j$ then the number of entries j in T_0 is less than φ and so $\frac{1}{\varphi!}e_{i,j}^{\varphi}\phi_{T_0} = 0$.

So assume that $j \le \mu_1$ and $\varphi \le \lambda_j$. Then we know there exists a λ-tableau T' in the above sum, and that T' is obtained from T by replacing φ entries j by i in row j. Since $i<j$ there is a column of T' in which two different squares contain the entry i. Hence $\phi_{T'} = 0$. So we have

$$\frac{y_{i,j}^{\varphi}}{\varphi!}\phi_\lambda = 0 \qquad \text{for all } \varphi \in \mathbf{N}.$$

Let $i,j \in I$ with $i<j$ and let $\chi \in \mathbf{N}$. Then

$$\frac{x_{i,j}^{\chi}}{\chi!}\phi_\lambda = \frac{(e_{i,\bar{j}}+e_{j,\bar{i}})^{\chi}}{\chi!}\phi_{T_0} = 0$$

since T_0 contains no barred entries. Similarly

$$\frac{z_i^{\zeta}}{\zeta!}\phi_\lambda = e_{i,\bar{i}}\phi_{T_0} = 0$$

for all $i \in I$ and all $\zeta \in \mathbb{N}$.

□

5.2.5 Lemma.

$$\mathcal{U}^{o}_{sp,\mathbf{Z}}\phi_\lambda \in \mathbf{Z}\phi_\lambda.$$

Proof. For convenience we use the notation h_i both for the element $h_{\alpha_i} \in \mathfrak{sp}_{2n}(\mathbf{k})$ and for its image in $\mathcal{U}^{o}_{sp,\mathbf{Z}}$ under the natural embedding.

Let $i \in \{1, \ldots, n-1\}$. Then $h_i = e_{i,i} - e_{\bar{i},\bar{i}} - e_{i+1,i+1} + e_{\overline{i+1},\overline{i+1}}$. The action of both $e_{\bar{i},\bar{i}}$ and $e_{\overline{i+1},\overline{i+1}}$ annihilates ϕ_λ, and so for any $\mu \in \mathbb{N}$,

$$\begin{aligned}
\binom{h_i}{\mu}\phi_\lambda &= \frac{h_i(h_i-1)\ldots(h_i-\mu+1)}{\mu!}\phi_\lambda \\
&= \frac{(e_{i,i}-e_{i+1,i+1})(e_{i,i}-e_{i+1,i+1}-1)\ldots(e_{i,i}-e_{i+1,i+1}-\mu+1)}{\mu!}\phi_\lambda \\
&= \frac{(\lambda_i-\lambda_{i+1})(\lambda_i-\lambda_{i+1}-1)\ldots(\lambda_i-\lambda_{i+1}-\mu+1)}{\mu!}\phi_\lambda
\end{aligned}$$

since $e_{k,k}\phi_\lambda = \lambda_k\phi_\lambda$ for all $k \in I$.

If $\mu > \lambda_i - \lambda_{i+1}$ then one of the terms in the numerator is equal to zero, and $0 \in \mathbf{Z}\phi_\lambda$. So assume that $\mu \leq \lambda_i - \lambda_{i+1}$. Then

$$\begin{aligned}
\binom{h_i}{\mu}\phi_\lambda &= \frac{(\lambda_i-\lambda_{i+1})!}{\mu!\,(\lambda_i-\lambda_{i+1}-\mu)!}\phi_\lambda \\
&= \binom{\lambda_i-\lambda_{i+1}}{\mu}\phi_\lambda \in \mathbf{Z}\phi_\lambda.
\end{aligned}$$

It remains to show $\binom{h_n}{\mu}\phi_\lambda \in \mathbf{Z}\phi_\lambda$ for all $\mu \in \mathbb{N}$. Now $h_n = e_{n,n} - e_{\bar{n},\bar{n}}$ and so $h_n\phi_\lambda = \lambda_n\phi_\lambda$. For any $\mu \in \mathbb{N}$

$$\binom{h_n}{\mu}\phi_\lambda = \frac{\lambda_n(\lambda_n-1)\ldots(\lambda_n-\mu+1)}{\mu!}\phi_\lambda.$$

This is zero if $\mu > \lambda_n$. Otherwise, $\mu \leq \lambda_n$ and

$$\binom{h_n}{\mu}\phi_\lambda = \binom{\lambda_n}{\mu}\phi_\lambda \in \mathbf{Z}\phi_\lambda.$$

□

5.2.6 Corollary.

$$V^{sp}_{\lambda,\mathbf{Z}} = \mathcal{U}^{-}_{sp,\mathbf{Z}}\phi_\lambda \quad \textit{and}$$
$$V^{sp}_{\lambda,\mathbf{k}} = (\mathcal{U}^{-}_{sp,\mathbf{k}})(\overline{\phi}_\lambda)$$

where $\mathcal{U}^-_{sp,\mathbf{k}} = \mathbf{k} \otimes \mathcal{U}^-_{sp,\mathbf{Z}}$.

5.2.7 Definitions. We define an ordering on $I \cup \bar{I}$ which we shall use throughout this chapter, and refer to as the *usual ordering* in subsequent chapters, by

$$1 < \bar{1} < 2 < \bar{2} < \ldots < n < \bar{n}.$$

Let the *modulus* $|\ | : I \cup \bar{I} \to I$ be the map defined by

$$|i| = i \ \text{ for all } i \in I$$
$$|\bar{i}| = i \ \text{ for all } \bar{i} \in \bar{I}.$$

As before, let $\mathcal{T}$ denote the set of λ-tableaux with entries from $I \cup \bar{I}$. We define a height function $\mathrm{ht} : \mathcal{T} \to \mathbf{Z}$ which maps a λ-tableau $T \in \mathcal{T}$ to the sum of the moduli of its entries. This gives an equivalence relation on $\mathcal{T}$ by

$$T_1 \sim T_2 \Leftrightarrow \mathrm{ht}(T_1) = \mathrm{ht}(T_2).$$

This equivalence relation partitions $\mathcal{T}$ into equivalence classes, and we will write $[T]$ for the equivalence class containing T. There is a total ordering on the set of equivalence classes given by

$$[T_1] < [T_2] \Leftrightarrow \mathrm{ht}(T_1) < \mathrm{ht}(T_2).$$

For any $i \in I \cup \bar{I}$ let $N_i : \mathcal{T} \to \mathbf{Z}$ be the map sending $T \in \mathcal{T}$ to the total number of entries equal to i in T.

5.2.8 Example. Let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$. Then $\mathrm{ht}(T) = 5$ and $N_{\bar{2}}(T) = 1$.

5.2.9 Lemma. *Let $i, j \in I$ with $i < j$ and let $T \in \mathcal{T}$ satisfy that there is exactly one row of T that contains an entry or entries i, and no entries $\bar{j}$ occur in that row. Then for all $s \in \mathbf{N}$ such that $s \leq N_i(T)$*

$$\frac{a^s_{i,j}}{s!}\psi_T = \psi_{T'} + \sum_{T''} z_{T''}\psi_{T''} \qquad z_{T''} \in \mathbf{Z}$$

where T' is the unique λ-tableau which is a row permutation of a tableau obtained from T by replacing s entries i by $\bar{j}$, which is non-decreasing along rows, and where the second sum involves only semistandard λ-tableaux T'' satisfying $[T''] < [T']$.

Proof.

$$\begin{aligned}
\frac{a_{\bar{i},j}^{s}}{s!}\psi_T &= \frac{(e_{\bar{i},j}+e_{\bar{j},i})^s}{s!}\psi_T \\
&= \sum_{t=0}^{s}\binom{s}{t}\frac{1}{s!}e_{\bar{i},j}^{t}e_{\bar{j},i}^{s-t}\psi_T \\
&= \frac{1}{s!}e_{\bar{j},i}^{s}\psi_T + \sum_{t=1}^{s}\frac{1}{t!\,(s-t)!}e_{\bar{i},j}^{t}e_{\bar{j},i}^{s-t}\psi_T.
\end{aligned}$$

By Lemma 5.2.3, $\frac{1}{s!}e_{\bar{j},i}^{s}\psi_T = \psi_{T'}$ as required. Since T' is obtained from T by s replacements of i by $\bar{j}$ we have $\mathrm{ht}(T') = s(j-i) + \mathrm{ht}(T)$.

Fix $t \in \{1, \ldots, s\}$. Since $\psi_T \in V_{\lambda,\mathbf{Z}}^{gl}$ for all $T \in \mathcal{T}$ and $\frac{e_{\bar{i},j}^{t}}{t!}\frac{e_{\bar{j},i}^{s-t}}{(s-t)!} \in \mathcal{U}_{\mathbf{Z}}^{\mathfrak{gl}}$ we have

$$\frac{e_{\bar{i},j}^{t}}{t!}\frac{e_{\bar{j},i}^{s-t}}{(s-t)!}\psi_T \in V_{\lambda,\mathbf{Z}}^{gl}.$$

If $N_j(T) < t$ then

$$\frac{e_{\bar{i},j}^{t}}{t!}\frac{e_{\bar{j},i}^{s-t}}{(s-t)!}\psi_T = 0.$$

Otherwise, by Lemma 5.2.1, it is equal to a sum of elements ϕ_{T^*} where T^* is a row permutation of a tableau obtained from T by replacing t entries j by $\bar{i}$ and $(s-t)$ entries i by $\bar{j}$. So $\mathrm{ht}(T^*) = \mathrm{ht}(T) + (s-2t)(j-i) < \mathrm{ht}(T')$. Also, by the proof of Lemma 3.3 in Carter and Lusztig [1], the sum of all such terms ϕ_{T^*} can be expressed as an integer combination of elements $\psi_{T''}$ where each T'' is a permutation of one of the T^*. Hence $\mathrm{ht}(T'') = \mathrm{ht}(T^*) < \mathrm{ht}(T')$ for all T''.

□

5.2.10 Example. Let $\lambda = (2,2)$ and $T = \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 & 2 \\\hline\end{array}$. Then

$$\begin{aligned}
\frac{a_{1,2}^2}{2}\psi_T &= \frac{a_{1,2}^2}{2}\phi_T \\
&= \frac{1}{2}(e_{\bar{2},1}+e_{\bar{1},2})^2\phi_T \\
&= \frac{1}{2}(e_{\bar{2},1}+e_{\bar{1},2})\left(\phi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline 2 & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline \bar{2} & 1 \\\hline 2 & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 & \bar{1} \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & 1 \\\hline \bar{1} & 2 \\\hline\end{array}}\right) \\
&= \frac{1}{2}\left\{\phi_{\begin{array}{|c|c|}\hline \bar{2} & \bar{2} \\\hline 2 & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline 2 & \bar{1} \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline \bar{1} & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline \bar{2} & \bar{2} \\\hline 2 & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline \bar{2} & 1 \\\hline 2 & \bar{1} \\\hline\end{array}}\right. \\
&\qquad +\phi_{\begin{array}{|c|c|}\hline \bar{2} & 1 \\\hline \bar{1} & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline 2 & \bar{1} \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline \bar{2} & 1 \\\hline 2 & \bar{1} \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & 1 \\\hline \bar{1} & \bar{1} \\\hline\end{array}} \\
&\qquad \left. +\phi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline \bar{1} & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline \bar{2} & 1 \\\hline \bar{1} & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & 1 \\\hline \bar{1} & \bar{1} \\\hline\end{array}}\right\} \\
&= \phi_{\begin{array}{|c|c|}\hline \bar{2} & \bar{2} \\\hline 2 & 2 \\\hline\end{array}} \\
&\qquad +\left(\phi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline 2 & \bar{1} \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline \bar{1} & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline \bar{2} & 1 \\\hline 2 & \bar{1} \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline \bar{2} & 1 \\\hline \bar{1} & 2 \\\hline\end{array}}+\phi_{\begin{array}{|c|c|}\hline 1 & 1 \\\hline \bar{1} & \bar{1} \\\hline\end{array}}\right) \\
&= \psi_{\begin{array}{|c|c|}\hline \bar{2} & \bar{2} \\\hline 2 & 2 \\\hline\end{array}}+\left(\psi_{\begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline \bar{1} & 2 \\\hline\end{array}}+\psi_{\begin{array}{|c|c|}\hline 1 & 1 \\\hline \bar{1} & \bar{1} \\\hline\end{array}}\right) \\
&= \psi_{\begin{array}{|c|c|}\hline \bar{2} & \bar{2} \\\hline 2 & 2 \\\hline\end{array}}+\left(-\psi_{\begin{array}{|c|c|}\hline 1 & \bar{1} \\\hline 2 & \bar{2} \\\hline\end{array}}-\psi_{\begin{array}{|c|c|}\hline 1 & 2 \\\hline \bar{1} & \bar{2} \\\hline\end{array}}+\psi_{\begin{array}{|c|c|}\hline 1 & 1 \\\hline \bar{1} & \bar{1} \\\hline\end{array}}\right).
\end{aligned}$$

5.2.11 Lemma. *Let $i, j \in I$ with $i < j$. Let $T \in \mathcal{T}$ satisfy that there is exactly one row containing an entry or entries i and no entries j occur in that row. For all $s \in \mathbf{N}$ such that $s \leq N_i(T)$*

$$\frac{b_{i,j}^s}{s!}\psi_T = \psi_{T'} + \sum_{T''} z_{T''}\psi_{T''} \qquad z_{T''} \in \mathbf{Z}$$

where T' is the unique λ-tableau which is non-decreasing along rows and is a row permutation of a tableau obtained from T by replacing s entries i by j, and where the sum over T'' involves only semistandard λ-tableaux T'' which satisfy $[T''] < [T']$.

Proof.

$$\begin{aligned}\frac{b_{i,j}^s}{s!}\psi_T &= \frac{(e_{j,i} - e_{\bar{i},\bar{j}})^s}{s!}\psi_T \\ &= \frac{1}{s!}e_{j,i}^s\psi_T + \sum_{t=1}^{s}\binom{s}{t}\frac{1}{s!}(-1)^t e_{j,i}^{s-t}e_{\bar{i},\bar{j}}^t\psi_T.\end{aligned}$$

By Lemma 5.2.3, $\frac{e_{j,i}^s}{s!}\psi_T = \psi_{T'}$ as required.

Let $t \in \{1, \ldots, s\}$. Then

$$\binom{s}{t}\frac{1}{s!}(-1)^t e_{j,i}^{s-t}e_{\bar{i},\bar{j}}^t\psi_T = (-1)^t\frac{e_{j,i}^{s-t}}{(s-t)!}\frac{e_{\bar{i},\bar{j}}^t}{t!}\psi_T$$

and by Lemma 5.2.1, this is equal to a sum of terms $\pm\phi_{T^*}$, where T^* is obtained from a row permutation of T by replacing t entries $\bar{j}$ by $\bar{i}$ and $(s-t)$ entries i by j. Since this sum is in $V_{\lambda,\mathbf{Z}}^{gl}$, by Lemma 3.3 of Carter and Lusztig [1], it is equal to an integer combination of elements $\psi_{T''}$, where the T'' are permutations of the T^*. Each T^* satisfies

$$\mathrm{ht}(T^*) = \mathrm{ht}(T) + (s-2t)(j-i) < \mathrm{ht}(T'),$$

hence each such T'' satisfies $[T''] < [T']$.

□

5.2.12 Definition. Let T be a symplectic λ-tableau. For all $i \in I$ define

$$\begin{aligned}\alpha_{i,j} &= \text{ number of entries } \bar{j} \text{ in row } i \text{ of } T \text{ for all } j \in I \text{ with } j > i;\\ \beta_{i,j} &= \text{ number of entries } j \text{ in row } i \text{ of } T \text{ for all } j \in I \text{ with } j > i;\\ \gamma_i &= \text{ number of entries } \bar{i} \text{ in row } i \text{ of } T.\end{aligned}$$

Define the element $v_T \in V_{\lambda,\mathbf{Z}}^{sp}$ by

$$v_T = \left(\prod_{\substack{i,j\in I\\ i<j}} \frac{a_{i,j}^{\alpha_{i,j}}}{\alpha_{i,j}!}\frac{b_{i,j}^{\beta_{i,j}}}{\beta_{i,j}!}\right)\left(\prod_{i\in I}\frac{c_i^{\gamma_i}}{\gamma_i!}\right)\phi_\lambda$$

where the left-hand product is in lexicographic order; i.e.

$$(1,1) < (1,2) < \ldots < (1,n) < (2,3) < \ldots < (n-1,n).$$

Note that $v_{T_0} = \phi_\lambda$. Let $\overline{v}_T$ be the image of v_T in $V^{sp}_{\lambda,\mathbf{k}}$, that is,

$$\overline{v}_T = 1 \otimes v_T$$
$$\overline{v}_T = \left[1 \otimes \left(\prod_{\substack{i,j \in I \\ i<j}} \frac{a_{i,j}^{\alpha_{i,j}}}{\alpha_{i,j}!} \frac{b_{i,j}^{\beta_{i,j}}}{\beta_{i,j}!} \right) \left(\prod_{i \in I} \frac{c_i^{\gamma_i}}{\gamma_i!} \right) \right] \overline{\phi}_\lambda.$$

5.2.13 Lemma. *Let T be a symplectic λ-tableau. Then*

$$v_T = \psi_T + \sum_{T'} c_{T'} \psi_{T'} \qquad \textit{for some } c_{T'} \in \mathbf{Z},$$

where the sum is over all semistandard λ-tableaux T' and $c_{T'} \neq 0 \Rightarrow [T'] < [T]$.

Proof. Since $\phi_\lambda = \psi_{T_0}$ where T_0 is the λ-tableau which has the entry i in all squares in the i^{th} row, for all i, repeated use of Lemma 5.2.3, gives

$$\prod_{i \in I} \frac{c_i^{\gamma_i}}{\gamma_i!} \phi_\lambda = \psi_{T_n}$$

where T_n has γ_i entries $\overline{i}$ and $(\lambda_i - \gamma_i)$ entries i in the i^{th} row for all i.

The next elements to act are $\frac{a_{n-1,n}^{\alpha_{n-1,n}}}{\alpha_{n-1,n}!} \frac{b_{n-1,n}^{\beta_{n-1,n}}}{\beta_{n-1,n}!}$. This is the identity if $\lambda_{n-1} = 0$. Otherwise row $n-1$ of T_n contains only entries equal to $n-1$ and $\overline{n-1}$. Furthermore, there are no entries equal to $n-1$ anywhere else in T_n. By using Lemmas 5.2.11and 5.2.9we have

$$\frac{a_{n-1,n}^{\alpha_{n-1,n}}}{(\alpha_{n-1,n})!} \frac{b_{n-1,n}^{\beta_{n-1,n}}}{(\beta_{n-1,n})!} \psi_{T_n} = \psi_{T_{n-1}} + \sum_{T'_{n-1}} \pm \psi_{T'_{n-1}}$$

where T_{n-1} is the λ-tableau containing γ_i entries $\overline{i}$ and $(\lambda_i - \gamma_i)$ entries i in row i for $i \in \{1, \dots, n-2\}$ and which is identical to T in rows $n-1$ and n, and where the sum is over a collection of semistandard λ-tableaux T'_{n-1} satisfying $[T'_{n-1}] < [T_{n-1}]$.

If $\lambda_{n-2} \neq 0$ then, in particular, row $n-2$ of T_{n-1} contains only entries equal to $n-2$ and $\overline{n-2}$, and there are no entries $n-2$ in any other row.

The next elements to act are

$$\frac{a_{n-2,n-1}^{\alpha_{n-2,n-1}}}{(\alpha_{n-2,n-1})!} \frac{b_{n-2,n-1}^{\beta_{n-2,n-1}}}{(\beta_{n-2,n-1})!} \frac{a_{n-2,n}^{\alpha_{n-2,n}}}{(\alpha_{n-2,n})!} \frac{b_{n-2,n}^{\beta_{n-2,n}}}{(\beta_{n-2,n})!}.$$

By using Lemmas 5.2.9and 5.2.11twice each we have

$$\left(\frac{a_{n-2,n-1}^{\alpha_{n-2,n-1}}}{(\alpha_{n-2,n-1})!}\frac{b_{n-2,n-1}^{\beta_{n-2,n-1}}}{(\beta_{n-2,n-1})!}\frac{a_{n-2,n}^{\alpha_{n-2,n}}}{(\alpha_{n-2,n})!}\frac{b_{n-2,n}^{\beta_{n-2,n}}}{(\beta_{n-2,n})!}\right)\left(\psi_{T_{n-1}}+\sum_{T'}\psi_{T'}\right)=\psi_{T_{n-2}}+\sum_{T'_{n-2}}\pm\psi_{T'_{n-2}}$$

where T_{n-2} contains γ_i entries equal $\bar{i}$ and $(\lambda_i-\gamma_i)$ entries i in row i for $i\in\{1,\ldots,n-3\}$ and is identical to T in rows $n-2$, ..., n, and where the sum is over a collection of semistandard λ-tableaux T'_{n-2} (possibly with multiplicity) such that $[T'_{n-2}]<[T_{n-2}]$.

Continuing in this way, after the whole product has acted on ϕ_λ, we obtain the required result.

□

5.2.14 Proposition. *The set $\{\bar{v}_T\ ;\ T\in\mathcal{T}$ is symplectic$\}$ is linearly independent over* **k**.

Proof. Assume the result is false. Then there exist distinct symplectic λ-tableaux $T_1,\ldots,$ T_s such that

$$c_{T_1}\bar{v}_{T_1}+\ldots+c_{T_s}\bar{v}_{T_s}=0$$

for some $c_{T_1},c_{T_2},\ldots,c_{T_s}\in\mathbf{k}\setminus\{0\}$.

Let $[T_i]$ be maximal amongst $[T_1],[T_2],\ldots,[T_s]$ according to the ordering on the equivalence classes of $\mathcal{T}$. Then $\bar{v}_{T_i}=\overline{\psi}_{T_i}+\sum_{T'_i}\pm\overline{\psi}_{T'_i}$ where the T'_i all satisfy $[T'_i]<[T_i]$. For all $j\in\{1,\ldots,s\}$ with $j\neq i$

$$\bar{v}_{T_j}=\overline{\psi}_{T_j}+\sum_{T'_j}\pm\overline{\psi}_{T'_j}$$

where $\overline{\psi}_{T_j}\neq\overline{\psi}_{T_i}$ and each T'_j satisfies $[T'_j]<[T_j]<[T_i]$ and hence $\overline{\psi}_{T'_j}\neq\overline{\psi}_{T_i}$. Since the elements $\overline{\psi}_T\in T^r(V)$ for all semistandard λ-tableaux T are linearly independent over **k** (see Carter and Lusztig [1] Theorem 3.5) we must have $c_{T_i}=0$, a contradiction. So the lemma is true.

□

5.3 The Spanning Property.

In Weyl [1] simple contraction operators were introduced in connection with finding irreducible modules of orthogonal and symplectic groups over a field of characteristic zero. In Wetherilt [1] the idea is extended to the case of an arbitrary infinite field in the context of finding Weyl modules for symplectic groups. More general contraction operators are needed to show that the elements $\overline{v}_T$, where T is a symplectic λ-tableau, span the module $V^{sp}_{\lambda,\mathbf{k}}$.

5.3.1 Definition. Define $T^r(\overline{V})_\lambda \subset T^r(\overline{V})$ to be the subspace spanned by tensors of the form $\overline{\phi}_T$ for some $T \in \mathcal{T}$.

Hence $T^r(\overline{V})_\lambda = \alpha T^r(\overline{V})$. The module $V^{gl}_{\lambda,\mathbf{k}}$ is a subspace of $T^r(\overline{V})_\lambda$ and so $V^{sp}_{\lambda,\mathbf{k}} \subset T^r(\overline{V})_\lambda$.

Contraction Operators.

We begin by considering the case when the partition diagram is a single column. Let $\gamma \in X(T)^+$ be of the form $\gamma = (1,1,\ldots,1,0,\ldots,0) = (1^m)$ for some $m \in \mathbb{N}$ with $m \leq n$. In this case $C(\gamma) = S_m$ and a basis for $T^m(\overline{V})_\lambda$ consists of tensors

$$\beta(\overline{v}_{i_1} \otimes \ldots \otimes \overline{v}_{i_m})$$

where $i_1 < \ldots < i_m$ and $\beta = \sum_{\sigma \in S_m} \mathrm{sign}(\sigma)\sigma$. Note that $\beta(\overline{v}_{i_1} \otimes \ldots \otimes \overline{v}_{i_m}) = \overline{\phi}_{\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_m \\ \hline \end{array}}$.

5.3.2 Definition. Let $t \in \mathbb{N}$ satisfy $2t \leq m$. Define the map $\omega_t : T^m(\overline{V})_\gamma \to T^{m-2t}(\overline{V})$ to be the linear extension of, for all $i_1, \ldots, i_m \in I \cup \overline{I}$ with $i_1 < \ldots < i_m$,

$$\omega_t(\beta(\overline{v}_{i_1} \otimes \ldots \otimes \overline{v}_{i_m})) = \beta' \sum_j \mathrm{sign}(\tau_j) < \overline{v}_{j_1}, \overline{v_{\overline{j_2}}} > \ldots < \overline{v_{j_{2t-1}}}, \overline{v_{\overline{j_{2t}}}} > \overline{v_{k_1}} \otimes \ldots \otimes \overline{v_{k_{m-2t}}}$$

where $\beta' = \sum_{\sigma \in S_{m-2t}} \mathrm{sign}(\sigma)\sigma$, and the sum is over all $j_1, \ldots, j_{2t} \in I(n, 2t)$ such that
(i) $\{j_1, j_2, \ldots, j_{2t}\} \subset \{i_1, \ldots, i_m\}$;
(ii) $j_1 < j_2 < \ldots < j_{2t}$;
and where

$$\{k_1, \ldots, k_{m-2t}\} = \{i_1, \ldots, i_m\} \setminus \{j_1, \ldots, j_{2t}\}$$

such that $k_1 < k_2 < \ldots < k_{m-2t}$, and τ_j is the permutation

$$\begin{pmatrix} i_1 & i_2 & \ldots & i_{m-2t} & i_{m-2t+1} & \ldots & i_m \\ k_1 & k_2 & \ldots & k_{m-2t} & j_1 & \ldots & j_{2t} \end{pmatrix}.$$

5.3.3 Example. Let $m = 3$ and $t = 1$. Then

$$\begin{aligned} \omega_1(\beta(\overline{v}_2 \otimes \overline{v}_1 \otimes \overline{v}_{\overline{2}})) &= -\omega_1(\beta(\overline{v}_1 \otimes \overline{v}_2 \otimes \overline{v}_{\overline{2}})) \\ &= -\beta' \left\{ < \overline{v}_1, \overline{v}_2 > \overline{v}_{\overline{2}} - < \overline{v}_1, \overline{v}_{\overline{2}} > \overline{v}_2 + < \overline{v}_2, \overline{v}_{\overline{2}} > \overline{v}_1 \right\} \\ &= -\beta' \overline{v}_1 \\ &= -\overline{v}_1. \end{aligned}$$

5.3.4 Example. Let $m = 5$ and $t = 2$. Then

$$\omega_1\left(\overline{\phi}_{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \overline{2} \\ \hline 3 \\ \hline \overline{3} \\ \hline \end{array}}\right) = \beta' \left\{ \begin{array}{l} < \overline{v_1}, \overline{v_2} >< \overline{v_{\overline{2}}}, \overline{v_3} > \overline{\phi}_{\boxed{\overline{3}}} - < \overline{v_1}, \overline{v_2} >< \overline{v_{\overline{2}}}, \overline{v_{\overline{3}}} > \overline{\phi}_{\boxed{3}} \\ + < \overline{v_1}, \overline{v_2} >< \overline{v_3}, \overline{v_{\overline{3}}} > \overline{\phi}_{\boxed{\overline{2}}} - < \overline{v_1}, \overline{v_{\overline{2}}} >< \overline{v_3}, \overline{v_{\overline{3}}} > \overline{\phi}_{\boxed{2}} \\ + < \overline{v_2}, \overline{v_{\overline{2}}} >< \overline{v_3}, \overline{v_{\overline{3}}} > \overline{\phi}_{\boxed{1}} \end{array} \right\}$$

$$= \overline{\phi}_{\boxed{1}}.$$

5.3.5 Notation. The following notations are used for the remainder of this chapter.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a sequence of elements from $I \cup \overline{I}$.

Let T_α be the tableau $\begin{array}{|c|}\hline \alpha_1 \\ \hline \vdots \\ \hline \alpha_m \\ \hline \end{array}$.

Let $\beta \prec_{2t} \alpha$ denote that β is a reordered subsequence of α, of length $2t$, such that $\beta_1 < \beta_2 < \ldots < \beta_{2t}$. Note that α need not be in increasing order.

Let $\alpha \setminus \beta$ be the sequence obtained by removing the elements of β and leaving the remaining elements in the same order as found in α.

For any $\beta \prec_{2t} \alpha$ define

$$< v_\beta > = < \overline{v_{\beta_1}}, \overline{v_{\beta_2}} >< \overline{v_{\beta_3}}, \overline{v_{\beta_4}} > \ldots < \overline{v_{\beta_{2t-1}}}, \overline{v_{\beta_{2t}}} > .$$

Since $\beta_1 < \beta_2 < \ldots < \beta_{2t}$ then this product is zero unless it is of the form

$$< \overline{v_{i_1}}, \overline{v_{\overline{i_1}}} >< \overline{v_{i_2}}, \overline{v_{\overline{i_2}}} > \ldots < \overline{v_{i_t}}, \overline{v_{\overline{i_t}}} >$$

for some finite sequence $i_1 < i_2 < \ldots < i_t$, in which case it is 1.

Given $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_{2t}) \prec_{2t} \alpha$. Let $\gamma = \alpha \setminus \beta$ and define

$$\tau_\beta^\alpha = \begin{pmatrix} \alpha_1 & \ldots & \alpha_{m-2t} & \alpha_{m-2t+1} & \ldots & \alpha_m \\ \gamma_1 & \ldots & \gamma_{m-2t} & \beta_1 & \ldots & \beta_{2t} \end{pmatrix}.$$

Let u be an element of $\mathcal{U}_{\mathbf{Z}}^{sp}$. Then $1 \otimes u \in \mathcal{U}_{\mathbf{k}}^{sp}$, and henceforth, by an abuse of notation, we will write u in place of $1 \otimes u$ for short.

We first state and prove the easiest of three lemmas.

5.3.6 Lemma. *For all $i \in I$ and all $m, s, t \in \mathbf{N}$ with $2t \leq m$ the map*

$$\omega_t : T^m(\overline{V})_{(1^m)} \to T^{m-2t}(\overline{V})$$

commutes with the action of $\frac{1}{s!}c_i^s \in \mathcal{U}_{sp,\mathbf{k}}^-$.

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a sequence of m distinct elements of $I \cup \bar{I}$. Then

$$\frac{1}{s!}c_i^s\overline{\phi_{T_\alpha}} = \overline{\phi_{T_{\alpha'}}}$$

where α' is obtained from α by replacing s entries i by $\bar{i}$. So $\frac{1}{s!}c_i^s\overline{\phi_{T_\alpha}} = 0$, unless $s = 1$ and α contains an entry equal to i.

Suppose $i \in \{\alpha_1, \ldots, \alpha_m\}$ and $s = 1$. Then let α^i be the sequence obtained from α by replacing the unique occurrence of i by $\bar{i}$.

First consider the case when $i, \bar{i} \in \{\alpha_1, \ldots, \alpha_m\}$. Then $c_i\overline{\phi_{T_\alpha}} = 0$ because if α^i is obtained from α by replacing i by $\bar{i}$ then α^i contains two equal entries, and $\overline{\phi_{T_{\alpha^i}}} = 0$. So $\omega_t c_i\overline{\phi_{T_\alpha}} = 0$. However

$$c_i\omega_t\overline{\phi_{T_\alpha}} = \sum_{\beta \prec_{2t} \alpha} \operatorname{sign}(\tau_\beta^\alpha) < v_\beta > c_i\overline{\phi_{T_\gamma}}$$

where $\gamma = \alpha \setminus \beta$. If $i \in \{\beta_1, \ldots, \beta_{2t}\}$ $i \notin \{\gamma_1, \ldots, \gamma_{m-2t}$ and so $c_i\overline{\phi_{T_\gamma}} = 0$. So only those $\beta \prec_{2t} \alpha$ which do not contain i, need be considered. If β contains $\bar{i}$ but not i then $< v_\beta >= 0$. So the sum need only be taken over those $\beta \prec_{2t} \alpha$ which do not contain i or $\bar{i}$. Then $c_i\overline{\phi_{T_\gamma}} = \overline{\phi_{T_{\gamma'}}}$ where γ' is obtained from γ by replacing i by $\bar{i}$. But then γ' has two equal entries and so $\overline{\phi_{\gamma'}} = 0$. Therefore the actions of ω_t and c_i commute in this case.

Secondly consider the case when $i \in \{\alpha_1, \ldots, \alpha_m\}$ but $\bar{i} \notin \{\alpha_1, \ldots, \alpha_m\}$. Then

$$\omega_t c_i\overline{\phi_{T_\alpha}} = \sum_{\beta' \prec_{2t} \alpha^i} \operatorname{sign}(\tau_{\beta'}^{\alpha^i}) < v_{\beta'} > \overline{\phi_{T_{\gamma'}}}$$

where α^i is obtained from α by replacing i by $\bar{i}$ and where $\gamma' = \alpha^i \setminus \beta'$. If β' contains the entry $\bar{i}$ then $< v_{\beta'} >= 0$ since it cannot contain the entry i. So the sum need only be taken over the set of $\beta' \prec_{2t} \alpha^i$ such that $\bar{i}$ is not in β'. However, this is equal to the set of $\beta \prec_{2t} \alpha$ which do not contain i. Thus the sum can be rewritten as

$$\omega_t c_i\overline{\phi_{T_\alpha}} = \sum_{\substack{\beta \prec_{2t} \alpha \\ i \notin \{\beta_1, \ldots, \beta_{2t}\}}} \operatorname{sign}(\tau_\beta^{\alpha^i}) < v_\beta > \overline{\phi_{T_{\gamma^i}}} \tag{A}$$

where γ^i is obtained from $\alpha \setminus \beta$ by replacing i by $\bar{i}$. However,

$$c_i\omega_t\overline{\phi_{T_\alpha}} = \sum_{\beta \prec_{2t} \alpha} \operatorname{sign}(\tau_\beta^\alpha) < v_\beta > c_i\overline{\phi_{T_\gamma}} \tag{B}$$

where $\gamma = \alpha \setminus \beta$. If β contains i then $c_i\overline{\phi_{T_\gamma}} = 0$. So the sum need only be taken over all $\beta \prec_{2t} \alpha$ which do not contain i. Thus $c_i\overline{\phi_{T_\gamma}} = \overline{\phi_{T_{\gamma^i}}}$ where γ^i is obtained from $\alpha \setminus \beta$ by replacing i by $\bar{i}$.

Let $\beta \prec_{2t} \alpha$ not contain i. Then the position of the entry i in the permutation τ^{α}_{β} is the same as the position of the entry $\bar{i}$ in $\tau^{\alpha^i}_{\beta}$; otherwise the two permutations are equal. Thus $\operatorname{sign}(\tau^{\alpha}_{\beta}) = \operatorname{sign}(\tau^{\alpha^i}_{\beta})$ and the right hand sides of (A) and (B) are equal. Hence $\omega_t c_i \overline{\phi_{T_\alpha}} = c_i \omega_t \overline{\phi_{T_\alpha}}$.

□

The second of the three lemmas is

5.3.7 Lemma. *For all $i,j \in I$ such that $i < j$ and for all $m,s,t \in \mathbb{N}$ with $2t \leq m$ the map*

$$\omega_t : T^m(\overline{V})_{(1^m)} \to T^{m-2t}(\overline{V})$$

commutes with the action of $\frac{1}{s!}b^s_{i,j} \in \mathcal{U}^-_{Sp,\mathbf{k}}$.

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a sequence of m distinct elements of $I \cup \bar{I}$. Then

$$\frac{1}{s!}b^s_{i,j}\overline{\phi}_{T_\alpha} = \sum_{\alpha_{s_1}} (-1)^{s-s_1}\overline{\phi}_{T_{\alpha_{s_1}}}$$

where the sum is over all α_{s_1} obtained from α by replacing s_1 entries i by j and $s - s_1$ entries $\bar{j}$ by $\bar{i}$.

If the total number of entries in α equal to either i or $\bar{j}$ is less than s then the sum is zero. In particular this sum is zero wherever $s > 2$.

The action of ω_t gives

$$\omega_t \frac{1}{s!}b^s_{i,j}\overline{\phi_{T_\alpha}} = \sum_{s_1=0}^{s}(-1)^{s-s_1} \sum_{\substack{\beta' \prec_{2t} \alpha_{s_1} \\ \beta'_1 < \cdots < \beta'_{2t}}} \operatorname{sign}(\tau^{\alpha_{s_1}}_{\beta'}) < v_{\beta'} > \overline{\phi_{T_{\gamma'_{s_1}}}} \tag{A}$$

where $\gamma'_{s_1} = \alpha_{s_1} \setminus \beta'$

On the other hand,

$$\omega_t \overline{\phi_{T_\alpha}} = \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \cdots < \beta_{2t}}} \operatorname{sign}(\tau^{\alpha}_{\beta}) < v_\beta > \overline{\phi_{T_\gamma}}$$

where $\gamma = \alpha \setminus \beta$. The action of $\frac{1}{s!}b^s_{i,j}$ gives

$$\frac{1}{s!}b^s_{i,j}\omega_t\overline{\phi_{T_\alpha}} = \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \cdots < \beta_{2t}}} \operatorname{sign}(\tau^{\alpha}_{\beta}) < v_\beta > \sum_{s_1=0}^{s}(-1)^{s-s_1}\overline{\phi_{T_{\gamma_{s_1}}}} \tag{B}$$

where γ_{s_1} is obtained from γ by replacing s_1 entries i by j and $(s - s_1)$ entries $\bar{j}$ by $\bar{i}$.

Again, if there are less than a total of s entries equal to either i or $\bar{j}$ in α then the same it true for all subsequences γ and the equation is zero. So whenever $s > 2$ equation (B) is also zero.

There are four cases to consider:-

(1) $s = 1, i \in \{\alpha_1, \ldots, \alpha_m\}$ and $\bar{j} \notin \{\alpha_1, \ldots, \alpha_m\}$;
(2) $s = 1, i \notin \{\alpha_1, \ldots, \alpha_m\}$ and $\bar{j} \in \{\alpha_1, \ldots, \alpha_m\}$;
(3) $s = 1$ and $i, \bar{j} \in \{\alpha_1, \ldots, \alpha_m\}$;
(4) $s = 2$ and $i, \bar{j} \in \{\alpha_1, \ldots, \alpha_m\}$.

Consider case (1), i.e. assume that $s = 1$, $i \in \{\alpha_1, \ldots, \alpha_m\}$ and $\bar{j} \notin \{\alpha_1, \ldots, \alpha_m\}$. Equation (A) gives

$$\omega_t b^s_{i,j} \overline{\phi T_\alpha} = \sum_{\substack{\beta' \prec_{2t} \alpha_1 \\ \beta'_1 < \cdots < \beta'_{2t}}} \operatorname{sign}(\tau^{\alpha_1}_{\beta'}) < v_{\beta'} > \overline{\phi T_{\gamma'_1}}$$

where α_1 is the sequence obtained from α by replacing i by j and where $\gamma'_1 = \alpha_1 \setminus \beta'$. If $\beta' \prec_{2t} \alpha'$ contains an entry j then $< v_{\beta'} >= 0$, since there is no element $\bar{j}$ in α_1 and so neither in β'. So the sum need only be over $\beta' \prec_{2t} \alpha_1$ such that $j \notin \{\beta'_1, \ldots, \beta'_{2t}\}$.

Equation (B) gives

$$b^s_{i,j} \omega_t \overline{\phi T_\alpha} = \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \cdots < \beta_{2t}}} \operatorname{sign}(\tau^{\alpha}_{\beta}) < v_{\beta} > \overline{\phi T_{\gamma_1}}$$

where γ_1 is obtained from $\alpha \setminus \beta$ by replacing an entry i by j. If $i \in \{\beta_1, \ldots, \beta_{2t}\}$ then there is no such γ_1 and so that term is zero. Therefore the sum is over all $\beta \prec_{2t} \alpha$ which do not contain i. This is the same as the set of $\beta' \prec_{2t} \alpha_1$ which do not contain j, and furthermore, if $\gamma_1 = \alpha \setminus \beta$ then $\gamma_1 = \gamma'_1$ when $\gamma'_1 = \alpha_1 \setminus \beta'$ and $\beta' = \beta$. So the two sums are equal.

Case (2) follows from the above by interchanging $\bar{j}$ and i and interchanging $\bar{i}$ and j. An extra minus sign is required before each expression.

Consider case (3), i.e. $s = 1$ and $i, \bar{j} \in \{\alpha_1, \ldots, \alpha_m\}$. This case splits into four parts:-

(3i) $\bar{i}, j \notin \{\alpha_1, \ldots, \alpha_m\}$;
(3ii) $\bar{i} \in \{\alpha_1, \ldots, \alpha_m\}$ and $j \notin \{\alpha_1, \ldots, \alpha_m\}$;
(3iii) $\bar{i} \notin \{\alpha_1, \ldots, \alpha_m\}$ and $j \in \{\alpha_1, \ldots, \alpha_m\}$;
(3iv) $\bar{i}, j \in \{\alpha_1, \ldots, \alpha_m\}$.

Consider case (3i). Equation (A) gives

$$\omega_t \frac{1}{s!} b^s_{i,j} \overline{\phi T_\alpha} = \sum_{s_1=0}^{1} (-1)^{s-s_1} \sum_{\substack{\beta' \prec_{2t} \alpha_{s_1} \\ \beta'_1 < \cdots < \beta'_{2t}}} \operatorname{sign}(\tau^{\alpha_{s_1}}_{\beta'}) < v_{\beta'} > \overline{\phi T_{\gamma'_{s_1}}}$$

where α_{s_1} is obtained from α by replacing s_1 entries i by j and $1 - s_1$ entries $\bar{j}$ by $\bar{i}$ and $\gamma'_{s_1} = \alpha_{s_1} \setminus \beta'$. Thus

$$\omega_t b^s_{i,j} \overline{\phi T_\alpha} = \sum_{\substack{\beta^i \prec_{2t} \alpha^i \\ \beta^i_1 < \cdots < \beta^i_{2t}}} \operatorname{sign}(\tau^{\alpha^i}_{\beta^i}) < v_{\beta^i} > \overline{\phi T_{(\gamma^i)}} - \sum_{\substack{\beta^{\bar{j}} \prec_{2t} \alpha^{\bar{j}} \\ \beta^{\bar{j}}_1 < \cdots < \beta^{\bar{j}}_{2t}}} \operatorname{sign}(\tau^{\alpha^{\bar{j}}}_{\beta^{\bar{j}}}) < v_{\beta^{\bar{j}}} > \overline{\phi T_{(\gamma^{\bar{j}})}}$$

where α^i and $\alpha^{\bar{j}}$ are the sequences obtained from α by replacing i by j and replacing $\bar{j}$ by $\bar{i}$ respectively, and where $\gamma^i = \alpha^i \setminus \beta^i$ and and $\gamma^{\bar{j}} = \alpha^{\bar{j}} \setminus \beta^{\bar{j}}$.

Let B_1 be the set of $\beta^i \prec_{2t} \alpha^i$ which contain j and $\bar{j}$. Let B_2 be the set of $\beta^{\bar{j}} \prec_{2t} \alpha^{\bar{j}}$ which contain i and $\bar{i}$. There is a bijection between B_1 and B_2 where $\beta_1^i \in B_1$ corresponds to $\beta_2^{\bar{j}} \in B_2$ if $\alpha^i \setminus \beta_1^i = \alpha^{\bar{j}} \setminus \beta_2^{\bar{j}}$, as sets, that is, up to reordering β_1^i and $\beta_2^{\bar{j}}$ are identical apart from j, $\bar{j}$ and i, $\bar{i}$ respectively.

Suppose $\beta_1^i \in B_1$ and $\beta_2^{\bar{j}} \in B_2$ are related in this way. Then $< v_{\beta_1^i} > = < v_{\beta_2^{\bar{j}}} >$, since $< \overline{v_i}, \overline{v_{\bar{i}}} > = < \overline{v_j}, \overline{v_{\bar{j}}} >$. Now $\beta_1^i = \sigma(i\ j)(\bar{i}\ \bar{j})\beta_2^{\bar{j}}$ where σ moves the pair j and $\bar{j}$ to be in increasing order amongst the entries of $(i\ j)(\bar{i}\ \bar{j})\beta_2^{\bar{j}}$. Since σ is permuting elements in adjacent pairs it is even. So

$$\tau_{\beta_1^i}^{\alpha^i} = \sigma(i\ j)(\bar{i}\ \bar{j})\tau_{\beta_2^{\bar{j}}}^{\alpha^{\bar{j}}}$$

and $\operatorname{sign}(\tau_{\beta_1^i}^{\alpha^i}) = \operatorname{sign}(\tau_{\beta_2^{\bar{j}}}^{\alpha^{\bar{j}}})$. Therefore

$$\operatorname{sign}(\tau_{\beta_1^i}^{\alpha^i}) < v_{\beta_1^i} > \overline{\phi_{T_{\gamma_1^i}}} - \operatorname{sign}(\tau_{\beta_2^{\bar{j}}}^{\alpha^{\bar{j}}}) < v_{\beta_2^{\bar{j}}} > \overline{\phi_{T_{\gamma_2^{\bar{j}}}}} = 0,$$

where $\gamma_1^i = \alpha^i \setminus \beta_1^i$ and $\gamma_2^{\bar{j}} = \alpha^{\bar{j}} \setminus \beta_2^{\bar{j}}$. So the terms of $\omega_t \frac{1}{s!} b_{i,j}\phi_{T_\alpha}$ involving the subsets B_1 and B_2 cancel in pairs, and we are left with

$$\omega_t b_{i,j}^s \overline{\phi_{T_\alpha}} = \sum_{\beta_3^i \in B_3} \operatorname{sign}(\tau_{\beta_3^i}^{\alpha^i}) < v_{\beta_3^i} > \overline{\phi_{T_{(\gamma^i)}}} - \sum_{\beta_4^{\bar{j}} \in B_4} \operatorname{sign}(\tau_{\beta_4^{\bar{j}}}^{\alpha^{\bar{j}}}) < v_{\beta_4^{\bar{j}}} > \overline{\phi_{T_{(\gamma^{\bar{j}})}}}$$

where

$$B_3 = \{\beta \prec_{2t} \alpha^i; \beta \text{ contains neither } j \text{ nor } \bar{j}\} \text{ and}$$
$$B_4 = \{\beta \prec_{2t} \alpha^{\bar{j}}; \beta \text{ contains neither } i \text{ nor } \bar{i}\}.$$

We do not need to consider any others, since if β contains only one of a pair $i, \bar{i}$ or $j, \bar{j}$ then $< v_\beta > = 0$. However, B_3 and B_4 are both equal to the set of all $\beta \prec_{2t} \alpha$ which contain neither i nor $\bar{j}$. So the equation can be written

$$\omega_t b_{i,j}^s \overline{\phi_{T_\alpha}} = \sum_{\beta \in B} \left(\operatorname{sign}(\tau_\beta^{\alpha^i}) < v_\beta > \overline{\phi_{T_{(\gamma)^i}}} - \operatorname{sign}(\tau_\beta^{\alpha^{\bar{j}}}) < v_\beta > \overline{\phi_{T_{(\gamma)^{\bar{j}}}}}\right)$$

where $(\gamma)^i$ is obtained from $\alpha \setminus \beta$ by replacing i by j and $(\gamma)^{\bar{j}}$ is obtained from $\alpha \setminus \beta$ by replacing $\bar{j}$ by $\bar{i}$, and B is the set of $\beta \prec_{2t} \alpha$ such that β does not contain i or $\bar{j}$. For all $\beta \in B$ we have $i \notin \{\beta_1, \ldots, \beta_{2t}\}$ and so $\operatorname{sign}(\tau_\beta^{\alpha^i}) = \operatorname{sign}(\tau_\beta^\alpha)$. Also $\bar{j} \notin \{\beta_1, \ldots, \beta_{2t}\}$ and so $\operatorname{sign}(\tau_\beta^{\alpha^{\bar{j}}}) = \operatorname{sign}(\tau_\beta^\alpha)$. Thus

$$\omega_t b_{i,j}^s \overline{\phi_{T_\alpha}} = \sum_{\beta \in B} \operatorname{sign}(\tau_\beta^\alpha) < v_\beta > \left(\overline{\phi_{T_{(\gamma)^i}}} - \overline{\phi_{T_{(\gamma)^{\bar{j}}}}}\right) \qquad (*)$$

Equation (B) gives

$$b^s_{i,j}\omega_t\overline{\phi T_\alpha} = \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \cdots < \beta_{2t}}} \operatorname{sign}(\tau^\alpha_\beta) \sum_{s_1=0}^{s} (-1)^{1-s_1} \overline{\phi_{T_{\gamma^{s_1}}}}$$

where γ_{s_1} is the sequence obtained from $\alpha \setminus \beta$ by replacing s_1 entries i by j and $1 - s_1$ entries $\bar{j}$ by $\bar{i}$. So this gives

$$b^s_{i,j}\omega_t\overline{\phi T_\alpha} = \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \cdots < \beta_{2t}}} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > \left(\overline{\phi_{T_{\gamma^i}}} - \overline{\phi_{T_{\gamma^{\bar{j}}}}}\right)$$

where γ^i is the sequence obtained from $\alpha \setminus \beta$ by replacing i by j and $\gamma^{\bar{j}}$ is the sequence obtained from $\alpha \setminus \beta$ by replacing $\bar{j}$ by $\bar{i}$.

Let $\beta \prec_{2t} \alpha$. If β contains either i or $\bar{j}$ then $< v_\beta >= 0$ since α, and hence β, does not contain either $\bar{i}$ or j. Thus the right-hand side of the above equation is equal to the right-hand side of equation $(*)$, and the lemma is true in this case.

Consider case (3ii). Assume that $i, \bar{i}, \bar{j} \in \{\alpha_1, \ldots, \alpha_m\}$ and $j \notin \{\alpha_1, \ldots, \alpha_m\}$. Since $\bar{i} \in \{\alpha_1, \ldots, \alpha_m\}$ then $e_{\bar{i},\bar{j}}\phi T_\alpha = 0$. Equation (A) gives

$$\omega_t b^s_{i,j}\overline{\phi T_\alpha} = \sum_{\substack{\beta^i \prec_{2t} \alpha^i \\ \beta^i_1 < \cdots < \beta^i_{2t}}} \operatorname{sign}(\tau^{\alpha^i}_{\beta^i}) < v_{\beta^i} > \overline{\phi_{T_{\gamma^i}}}$$

where α^i is obtained from α by replacing i by j and $\gamma^i = \alpha^i \setminus \beta^i$. If $\beta^i \prec_{2t} \alpha^i$ contains either j or $\bar{j}$, but not both, then $< v_{\beta^i} >= 0$, and can be removed from the sum.

Let B_1 be the set of $\beta^i \prec_{2t} \alpha^i$ which contain j and $\bar{j}$. Let B_2 be the set of $\beta^i \prec_{2t} \alpha^i$ which do not contain either j or $\bar{j}$. Then the set B_2 is equal to the set of all $\beta \prec_{2t} \alpha$ which do not contain i or $\bar{j}$, and hence we can assume do not contain $\bar{i}$ either (since otherwise $< v_\beta >= 0$). For all $\beta \in B_2$ τ^α_β and $\tau^{\alpha^i}_\beta$ are identical except that the entry i in τ^α_β is replaced by j in $\tau^{\alpha^i}_\beta$. In particular these permutations have the same sign and so the above equation can be rewritten as

$$\omega_t b^s_{i,j}\overline{\phi T_\alpha} = \sum_{\beta^1 \in B_1} \operatorname{sign}(\tau^{\alpha^i}_{\beta^1}) < v_{\beta^1} > \overline{\phi_{T_{\gamma^1}}} + \sum_{\beta^2 \in B_2} \operatorname{sign}(\tau^\alpha_{\beta^2}) < v_{\beta^2} > \overline{\phi_{T_{(\gamma^2)^i}}} \qquad (C)$$

where $\gamma^1 = \alpha^i \setminus \beta^1$ and $(\gamma^2)^i$ is obtained from $\alpha \setminus \beta^2$ by replacing i by j.

Equation (B) gives

$$b^s_{i,j}\omega_t\overline{\phi}T_\alpha = \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \cdots < \beta_{2t}}} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > \left(\overline{\phi}_{T_{\gamma^i}} - \overline{\phi}_{T_{\gamma^{\bar{j}}}}\right)$$

where γ^i is obtained from $\alpha \setminus \beta$ by replacing i by j and $\gamma^{\bar{j}}$ is obtained form $\alpha \setminus \beta$ by replacing $\bar{j}$ by $\bar{i}$.

If $\beta \prec_{2t} \alpha$ contains $\bar{\imath}$ but not i then $< v_\beta >= 0$, and can be removed from the sum without effect.

If β does not contain $\bar{\imath}$ then $\gamma^{\bar{\jmath}}$ contains two entries which equal i and so $\overline{\phi_{T_{\gamma^{\bar{\jmath}}}}} = 0$.

If β contains i then $\alpha \setminus \beta$ does not contain i and so there can be no γ^i obtained from $\alpha \setminus \beta$ by replacing i by j.

Define the sets B_3 and B_4 by

$$B_3 = \{\beta \prec_{2t} \alpha; \beta \text{ contains } i \text{ and } \bar{\imath}\}$$
$$B_4 = \{\beta \prec_{2t} \alpha; \beta \text{ contains neither } i \text{ nor } \bar{\imath}\}$$

Then the above equation gives

$$b^s_{i,j}\omega_t\overline{\phi_{T_\alpha}} = -\sum_{\beta \in B_3} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > \overline{\phi_{T_{\gamma^{\bar{\jmath}}}}} + \sum_{\beta \in B_4} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > \overline{\phi_{T_{\gamma^i}}}. \qquad (D)$$

Notice that B_4 is equal to B_2 and that for each $\beta \in B_2 = B_4$ $\phi_{T_{\gamma_2}} = \phi_{T_{\gamma^i}}$. So the second sums on the right-hand sides of equations (C) and (D) are equal.

If $\beta \in B_3$ contains $\bar{\jmath}$ then $< v_\beta >= 0$ because $j \notin \alpha$. So omit such β. Then for $\beta \in B_3$ $\alpha \setminus \beta$ contains an entry $\bar{\jmath}$ which is replaced by $\bar{\imath}$ to obtain $\gamma^{\bar{\jmath}}$.

If $\beta \in B_1$ such that β contains $\bar{\imath}$ then β does not contain i and so $< v_\beta >= 0$. So remove these β from B_1; this does not affect the sum in equation (D). There is now a bijection between B_1 and B_3, $\beta^1 \in B_1$ corresponding to $\beta^3 \in B_3$ when β^1 and β^3 have exactly the same entries except for the pair $j, \bar{\jmath}$ in β^1 and the pair $i, \bar{\imath}$ in β^3. For such β^1 and β^3 $< v_{\beta^1} >=< v_{\beta^3} >$ and $\gamma_1 = \alpha^i \setminus \beta^1$ is equal to $\gamma^{\bar{\jmath}}$, up to reordering, where $\gamma = \alpha \setminus \beta^3$ and $\gamma^{\bar{\jmath}}$ is obtained from γ by replacing $\bar{\jmath}$ by $\bar{\imath}$. Let σ_1 be the permutation satisfying $\sigma_1(\gamma_1) = \sigma^{\bar{\jmath}}$. Hence

$$\overline{\phi}_{T_{\gamma^{\bar{\jmath}}}} = \operatorname{sign}(\sigma_1)\overline{\phi}_{T_{\gamma_1}}$$

Let $(\beta^1)'$ be the sequence obtained by replacing the pair $j, \bar{\jmath}$ by $i, \bar{\imath}$, and let σ_2 be the permutation satisfying $\sigma_2(\beta^1)' = \beta^3$. Let $(\sigma_1)'$ be the permutation obtained from σ by replacing every occurrence of $\bar{\imath}$ by $\bar{\jmath}$. Then

$$(\sigma_1)'\sigma_2(\tau^{\alpha^i}_{\beta^1}) = \tau^\alpha_{\beta^3}$$

and since $\operatorname{sign}(\sigma) = \operatorname{sign}((\sigma)')$

$$\operatorname{sign}(\tau^\alpha_{\beta^3}) = -\operatorname{sign}(\sigma)\operatorname{sign}(\tau^{\alpha^i}_{\beta^1}).$$

Hence the two first sums on the right-hand sides of equations (C) and (D) are equal. This proves the lemma in this case.

Consider case (3iii). This is proved in an analogous way to case (3ii) by interchanging the rôles of i and $\bar{\jmath}$ and of $\bar{\imath}$ and j. An extra minus sign is also required before some of the expressions.

Consider case (3iv). Assume that $i, \bar{i}, j, \bar{j} \in \{\alpha_1, \dots, \alpha_m\}$. As α^i is obtained from α by replacing the entry i by j then T_{α^i} contains two equal entries in the same column. So $\overline{\phi_{T_{\alpha^i}}} = 0$ Similarly, if $\alpha_{\bar{j}}$ is obtained from α by replacing $\bar{j}$ by $\bar{i}$ then $\overline{\phi_{T_{\alpha^{\bar{j}}}}} = 0$. Therefore $b_{i,j}\overline{\phi_{T_\alpha}} = 0$ and

$$\omega_t b_{i,j}\overline{\phi_{T_\alpha}} = 0.$$

Alternatively,

$$b_{i,j}\omega_t\overline{\phi_{T_\alpha}} = \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \dots < \beta_{2t}}} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > e_{j,i}\overline{\phi_{T_\gamma}} - \sum_{\substack{\beta \prec_{2t} \alpha \\ \beta_1 < \dots < \beta_{2t}}} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > e_{\bar{i},\bar{j}}\overline{\phi_{T_\gamma}}.$$

In the first sum any sequences $\beta \prec_{2t} \alpha$ which do not contain j can be removed without affecting the sum since $e_{j,i}\overline{\phi_{T_\gamma}} = 0$ for such β. Those $\beta \prec_{2t} \alpha$, which do not contain both j and $\bar{j}$, can also be removed since otherwise $< v_\beta >= 0$ in this case. Therefore the first sum is over all $\beta \prec_{2t} \alpha$ which contain both j and $\bar{j}$. We can also omit from the sum any β containing either i $\bar{i}$ since then we have either $< V_\beta >= 0$ or $e_{j,i}\overline{\phi_{T_\gamma}} = 0$. Let the set of all such β be B_1. The first sum thus equals

$$\sum_{\beta \in B_1} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > \overline{\phi_{T_{\gamma^i}}}$$

where γ^i is obtained from $\alpha \setminus \beta$ by replacing i by j.

In the second sum any sequences $\beta \prec_{2t} \alpha$ which does not contain $\bar{i}$ can be removed from the sum since otherwise $\gamma = \alpha \setminus \beta$ contains $\bar{i}$ and so $e_{\bar{i},\bar{j}}\overline{\phi_{T_\gamma}} = 0$. We can also omit any β containing j or $\bar{j}$ since then either $< v_\beta >= 0$ or $e_{\bar{i},\bar{j}}\overline{\phi_{T_\gamma}} = 0$. Hence the sum is over all $\beta \prec_{2t} \alpha$ which contain both i and $\bar{i}$, but neither j nor $\bar{j}$. Let the set of all such β be denoted by B_2. Thus the second sum is equal to

$$\sum_{\beta \in B_2} \operatorname{sign}(\tau^\alpha_\beta) < v_\beta > \overline{\phi_{T_{\gamma^{\bar{j}}}}}$$

where $\gamma^{\bar{j}}$ is obtained from $\alpha \setminus \beta$ by replacing $\bar{j}$ by $\bar{i}$.

There is a bijection between the two sets B_1 and B_2; $\beta^1 \in B_1$ corresponds to $\beta^2 \in B_2$ if they are identical when the entries i and $\bar{i}$ are removed from β^1 and j and $\bar{j}$ are removed from β^2. For such $\beta^1 \in B_1$ and $\beta_2 \in B_2$ $< v_{\beta^1} >=< v_{\beta_2} >$ because $< \overline{v_i}, \overline{v_{\bar{i}}} >=< \overline{v_j}, \overline{v_{\bar{j}}} >$. So β^1 is identical to β^2 except that the former has the entries j and $\bar{j}$ instead of the entries i and $\bar{i}$ in the latter. So $\alpha \setminus \beta^1$ and $\alpha \setminus \beta^2$ are also identical up to reordering except that the former has the entries i and $\bar{i}$ instead of the entries j and $\bar{j}$ in the latter. Thus $\gamma^i_1 = (\alpha \setminus \beta^1)^i$ is identical to $\gamma^{\bar{j}}_2 = (\alpha \setminus \beta^2)^{\bar{j}}$ up to reordering. Let σ be the permutation satisfying $\sigma(\gamma^i_1) = \gamma^{\bar{j}}_2$. Then

$$\overline{\phi_{T_{\gamma^{\bar{j}}_2}}} = \operatorname{sign}(\sigma)\overline{\phi_{T_{\gamma^i_1}}}.$$

However, it can also be seen that

$$\operatorname{sign}(\tau^\alpha_{\beta^2}) = \operatorname{sign}(\sigma)\operatorname{sign}(\tau^\alpha_{\beta^1}).$$

Therefore the terms cancel in pairs, giving zero. Thus the right hand sides of equations (A) and (B) are equal and the lemma is true in this case.

Case (4) breaks down into the same four subcases as case (3). In each one we find expressions for $\frac{1}{2}b_{i,j}^2\omega_t\overline{\phi_{T_\alpha}}$ and $\omega_t\frac{1}{2}b_{i,j}^2\overline{\phi_{T_\alpha}}$ and show that some terms in these expressions are zero, some cancel in pairs, and the remaining terms are equal in both expressions. The calculations are simpler in this case, and we omit the details here.

□

The last of these three lemmas follows.

5.3.8 Lemma. *For all $i,j \in I$ such that $i < j$ and for all $m, s, t \in \mathbb{N}$ with $2t \leq m$ the map*

$$\omega_t : T^m(\overline{V})_{(1^m)} \to T^{m-2t}(\overline{V})$$

commutes with the action of $\frac{1}{s!}a_{i,j}^s \in \mathcal{U}_{Sp,\mathbf{k}}^-$.

Proof. This proof is along similar lines to the proof of the last lemma. Again we have four cases to consider

(1) $s = 1$, $i \in \{\alpha_1, \ldots, \alpha_m\}$ and $j \notin \{\alpha_1, \ldots, \alpha_m\}$;
(2) $s = 1$, $i \notin \{\alpha_1, \ldots, \alpha_m\}$ and $j \in \{\alpha_1, \ldots, \alpha_m\}$;
(3) $s = 1$ and $i, j \in \{\alpha_1, \ldots, \alpha_m\}$;
(4) $s = 2$ and $i, j \in \{\alpha_1, \ldots, \alpha_m\}$.

Cases (3) and (4) both break down into four subcases

(i) $\overline{i}, \overline{j} \notin \{\alpha_1, \ldots, \alpha_m\}$;
(ii) $\overline{i} \in \{\alpha_1, \ldots, \alpha_m\}$ and $\overline{j} \notin \{\alpha_1, \ldots, \alpha_m\}$;
(iii) $\overline{i} \notin \{\alpha_1, \ldots, \alpha_m\}$ and $\overline{j} \in \{\alpha_1, \ldots, \alpha_m\}$;
(iv) $\overline{i}, \overline{j} \in \{\alpha_1, \ldots, \alpha_m\}$.

Each of the above cases needs to be checked using techniques similar to those used in the previous two proofs. These involve finding expressions for $\frac{1}{s!}a_{i,j}^s\omega_t\phi_{T_\alpha}$ and for $\omega_t\frac{1}{s!}a_{i,j}^s\phi_{T_\alpha}$ and recognising that some terms in the expressions are zero, some cancel in pairs, and that the remaining non-zero terms are equal in both expressions. We omit the details.

□

5.3.9 Proposition. *For all $t, m \in \mathbb{N}$ with $2t \leq m$ the map $\omega_t : T^m(\overline{V})_{(1^m)} \to T^{m-2t}(\overline{V})$ and the action of $\mathcal{U}_{sp,\mathbf{k}}^-$ commute.*

Proof. Continuing to write u for $1 \otimes u$ in $\mathcal{U}_{sp,\mathbf{k}}^-$, we know that $\mathcal{U}_{sp,\mathbf{k}}^-$ is generated by elements of the form $\frac{1}{\alpha_{i,j}!}a_{i,j}^{\alpha_{i,j}}$, $\frac{1}{\beta_{i,j}!}b_{i,j}^{\beta_{i,j}}$ and $\frac{1}{\gamma_k!}c_k^{\gamma_k}$ for all $i, j \in I \cup \overline{I}$, with $i < j$, all $k \in I$, and for all non-negative integers $\alpha_{i,j}$, $\beta_{i,j}$, γ_k.

By Lemmas 5.3.6, 5.3.7 and 5.3.8 the actions of these elements commute with the map ω_t for all $t \in \mathbb{N}$ such that $2t \leq m$, and hence so does the action of $\mathcal{U}_{sp,\mathbf{k}}^-$.

□

Let T be a λ-tableau with columns $T_1, \ldots, T_{\lambda_1}$. Let μ be the conjugate partition to λ. The subgroup $C(\lambda) \subset S_r$ of all permutations which fix the columns of T satisfies

$$C(\lambda) \cong S_{\mu_1} \times \ldots \times S_{\mu_{\lambda_1}}.$$

By the associativity of tensor products

$$T^r(\overline{V})_\lambda = T^{\mu_1}(\overline{V})_{(1^{\mu_1})} \otimes T^{\mu_2}(\overline{V})_{(1^{\mu_2})} \otimes \ldots \otimes T^{\mu_{\lambda_1}}(\overline{V})_{(1^{\mu_{\lambda_1}})}.$$

5.3.10 Definition. Suppose $s \in \{1, \ldots, \lambda_1\}$ and $t \in \mathbb{N}$ satisfies $2t \leq \mu_s$. The *contraction operator* $\Omega_{s,t} : T^r(\overline{V})_\lambda \to T^{r-2t}(\overline{V})$ is given by

$$\Omega_{s,t} = 1 \otimes \ldots \otimes 1 \otimes \omega_t \otimes 1 \otimes \ldots \otimes 1$$

where $T^r(\overline{V})_\lambda$ is considered to be

$$T^{\mu_1}(\overline{V})_{(1^{\mu_1})} \otimes T^{\mu_2}(\overline{V})_{(1^{\mu_2})} \otimes \ldots \otimes T^{\mu_{\lambda_1}}(\overline{V})_{(1^{\mu_{\lambda_1}})}$$

and ω_t acts on the s^{th} term in the tensor product.

Since $\omega_t : T^{\mu_s}(\overline{V})_{(1^{\mu_s})} \to T^{\mu_s - 2t}(\overline{V})$ commutes with the action of $\mathcal{U}^-_{sp,\mathbf{k}}$, then $\Omega_{s,t}$ also commutes with the action of $\mathcal{U}^-_{sp,\mathbf{k}}$.

5.3.11 Definition. A tensor $v \in T^r(\overline{V})_\lambda$ is said to be *traceless* if $\Omega_{s,t}(v) = 0$ for all $s \in \{1, \ldots, \lambda_1\}$ and all t such that $2t \leq \mu_s$.

5.3.12 Proposition. *All the elements of $V^{sp}_{\lambda,\mathbf{k}}$ are traceless.*

Proof. Since the action of $\mathcal{U}^-_{sp,\mathbf{k}}$ commutes with all the contraction operators $\Omega_{s,t}$ and $V^{sp}_{\lambda,\mathbf{k}}$ is generated by $\overline{\phi}_\lambda$ under the action of $\mathcal{U}^-_{sp,\mathbf{k}}$ it is enough to show that $\overline{\phi}_\lambda$ is traceless.

$\overline{\phi}_\lambda = \overline{\phi}_{T_0}$, where T_0 is the basic λ-tableau. Let $T_0^1, \ldots, T_0^{\lambda_1}$ be the columns of T_0. Suppose $s \in \{1, \ldots, \lambda_1\}$ and $t \in \mathbb{N}$ satisfies $2t \leq \mu_s$. Then

$$\begin{aligned}\Omega_{s,t}\overline{\phi}_{T_0} &= \Omega_{s,t}(\overline{\phi}_{T_0^1} \otimes \ldots \otimes \overline{\phi}_{T_0^{\lambda_1}}) \\ &= \overline{\phi}_{T_0^1} \otimes \ldots \otimes \overline{\phi}_{T_0^{s-1}} \otimes \omega_t(\overline{\phi}_{T_0^s 1}) \otimes \overline{\phi}_{T_0^{s+1}} \otimes \ldots \otimes \overline{\phi}_{T_0^{\lambda_1}}.\end{aligned}$$

Now $\overline{\phi}_{T_0^s} = \sum_{\sigma \in S_{\mu_s}} \text{sign}(\sigma)\sigma(\overline{v}_1 \otimes \ldots \otimes \overline{v}_{\mu_s})$. and so $\overline{\phi}_{T_0^s}$ is a sum of tensors $\overline{v}_{i_1} \otimes \ldots \otimes \overline{v}_{i_{\mu_s}}$ for some $i_1, \ldots, i_{\mu_s} \in I$. Let $\{j_1, \ldots, j_{2t}\} \subset \{i_1, \ldots, i_{\mu_s}\}$. Then $j_1, \ldots, j_{2t} \in \{1, \ldots, \mu_s\}$ and

$$< \overline{v}_{j_1}, \overline{v}_{j_2} >< \overline{v}_{j_3}, \overline{v}_{j_4} > \ldots < \overline{v}_{j_{2t-1}}, \overline{v}_{j_{2t}} >= 0.$$

Thus $\omega_t(\overline{v}_{i_1} \otimes \ldots \otimes \overline{v}_{i_{\mu_s}}) = 0$, which implies $\omega_t \overline{\phi}_\lambda = 0$. Hence $\Omega_{s,t}\overline{\phi}_{T_0} = 0$.

□

5.3.13 Definitions. Suppose $s \in \{1, \ldots, \lambda_1\}$ and $t \in \mathbb{N}$ satisfies $2t \leq \mu_s$. A $\lambda_{s,t}$-diagram is a λ-diagram with the bottom $2t$ squares removed from the s^{th} column. A $\lambda_{s,t}$-tableau is a $\lambda_{s,t}$-diagram with entries from $I \cup \overline{I}$, one entry in each square.

Let T be a $\lambda_{s,t}$-tableau and let $T_1, \ldots, T_{\lambda_1}$ be its columns. Define $\overline{\phi}_T \in T^{r-2t}(\overline{V})$ by

$$\overline{\phi}_T = \overline{\phi}_{T_1} \otimes \ldots \otimes \overline{\phi}_{T_{s-1}} \otimes \sum_{\sigma \in S_{\mu_s - 2t}} \operatorname{sign}(\sigma)\sigma(v_{T_s(1)} \otimes \ldots \otimes v_{T_s(\mu_s - 2t)}) \otimes \overline{\phi}_{T_{s+1}} \otimes \ldots \otimes \overline{\phi}_{T_{\lambda_1}}$$

where $T_s(i)$ is the element in the i^{th} row of T_s. Let $T^{r-2t}(\overline{V})_{\lambda_{s,t}}$ denote the k-space generated by the vectors $\overline{\phi}_T$ for all $\lambda_{s,t}$-tableaux T.

Therefore

$$\Omega_{s,t} : T^r(\overline{V})_\lambda \to T^{r-2t}(\overline{V})_{\lambda_{s,t}}.$$

We will consider what the property of being traceless tells us about a vector in $T^r(\overline{V})_\lambda$.

Let $\overline{v} \in T^r(V)_\lambda$. Then $\overline{v}$ has a unique expression

$$\overline{v} = \sum_{T' \in \mathcal{T}} k_{T'} \overline{t}_{T'} \qquad \text{for some } k_{T'} \in \mathbf{k}$$

where t'_i denotes the entry in the i^{th} position of T' and $\overline{t}_{T'} = \overline{v}_{t'_1} \otimes \ldots \otimes \overline{v}_{t'_r}$. Suppose that $\overline{v}$ is traceless. Let $s \in \{1, \ldots, \lambda_1\}$ and $t \in \mathbb{N}$ satisfy $2t \leq \mu_s$. Let T be a $\lambda_{s,t}$-tableau. The coefficient of $\overline{t}_T$ in $\Omega_{s,t}(v)$ is

$$\sum_{\substack{i_1, \ldots, i_t \in I \\ i_1 < \ldots < i_t}} k_{T^s(i_1, \ldots, i_t)}$$

where $T^s(i_1, \ldots, i_t)$ denotes the λ-tableau obtained from T by adjoining $2t$ squares to the bottom of the s^{th} column and entering i_1, $\overline{i_1}$, i_2, $\overline{i_2}$, $\ldots$, i_t, $\overline{i_t}$ into them.

Since v is traceless, $\Omega_{s,t}(v) = 0$ and

$$\sum_{\substack{i_1, \ldots, i_t \in I \\ i_1 < \ldots < i_t}} k_{T^s(i_1, \ldots, i_t)} = 0. \tag{X}$$

These equations are satisfied for all $s \in \{1, \ldots, \lambda_1\}$, for all $t \in \mathbb{N}$ such that $2t \leq \mu_s$ and all $\lambda_{s,t}$-tableaux T.

Equations of this sort are called *compound relations* between the coefficients $k_{T'}$ of v. In the special case where $t = 1$ such an equation is called a *simple relation.*

As before let $\mathcal{T}$ denote the set of λ-tableaux with entries from $I \cup \overline{I}$. Consider functions $f : \mathcal{T} \to G$ where G is an abelian group, satisfying:-

(i) $f(T) = 0$ if T has equal entries in two distinct squares belonging to the same column;

(ii) whenever $T, T' \in \mathcal{T}$ satisfy that T' is obtained from T by interchanging two squares of the same column

$$f(T) + f(T') = 0;$$

(iii) for all $s \in \{1, \ldots, \lambda_1\}$, all $t \in \mathbb{N}$ such that $2t \leq \mu_s$ and all $\lambda_{s,t}$-tableaux Y

$$\sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} f(Y^s(i_1, \ldots, i_t)) = 0.$$

5.3.14 Lemma. *Suppose $f : \mathcal{T} \to G$ satisfies conditions (i), (ii) and (iii) above. Let $T \in \mathcal{T}$ be a semistandard λ-tableau which is not symplectic. Then $f(T)$ can be expressed as a linear combination of elements $f(T') \in G$ where each $T' \in \mathcal{T}$ is obtained from T by replacing a number of entries by others of greater modulus.*

Proof. Let $T \in \mathcal{T}$ be a semistandard λ-tableau which is not symplectic and denote the entries of T by $t_1, \ldots, t_r \in I \cup \overline{I}$, where t_i denotes the entry in the i^{th} position of T. Since T is not symplectic there must be some row $h \in \{1, \ldots, \mu_1\}$ with an entry of modulus less than h. Suppose this entry occurs in column $s \in \{1, \ldots, \lambda_1\}$. Let $I_s = \{i_1, \ldots, i_{\mu_s}\} \subset \{1, \ldots, r\}$ be the s^{th} column of the leading tableau as before. Let the entries in the s^{th} column of T be $t_{i_1}, \ldots, t_{i_{\mu_s}}$. Since T is semistandard $t_{i_1} < \ldots < t_{i_{\mu_s}}$. Thus $|t_{i_h}| < h$ and so $t_{i_1}, \ldots, t_{i_h} \in \{1, \overline{1}, \ldots, h-1, \overline{h-1}\}$. Define $R_h \subset I$ by

$$R_h = \left\{k \; ; \; k, \overline{k} \in \{t_{i_1}, \ldots, t_{i_h}\}\right\}.$$

Then R_h is the set of repeats up to row h. Let O_h be given by

$$O_h = \left\{k; \text{ neither } k \text{ nor } \overline{k} \text{ are in } \{t_{i_1}, \ldots, t_{i_h}\}\right\}.$$

Then O_h is the set of omitted moduli up to row h.

We claim that $|O_h| = |R_h| - 1$. Let $t = |R_h|$. Up to row h there are h squares to be filled in the s^{th} column of T, and entries can come from $\{1, \overline{1}, \ldots, h-1, \overline{h-1}\}$. The t repeated entries use up $2t$ squares, leaving $h - 2t$ squares to be filled. To fill these remaining squares, without repeated entries, there are $(h-1) - t = h - t - 1$ moduli of entries available. This leaves $(h - t - 1) - (h - 2t) = t - 1$ moduli less than h of which there will be no entry in column s of T. This proves the claim.

Write $R_h = \{r_1, \ldots, r_t\}$ and $O_h = \{o_1, \ldots, o_{t-1}\}$. Let $\sigma \in C(\lambda)$ be the permutation that sends the squares in column s of T containing $r_1, \overline{r}_1, \ldots, r_t$ and $\overline{r}_t$ to the bottom of the column, leaving the remaining squares in their original order. Since f satisfies condition (ii)

$$f(T) = \text{sign}(\sigma) f(\sigma(T)).$$

Let Y be the $\lambda_{s,t}$-tableau obtained from $\sigma(T)$ by removing the lowest $2t$ squares and their entries from column s. Then

$$Y^s(r_1, \ldots, r_t) = \sigma(T)$$

and so

$$f(Y^s(r_1,\ldots,r_t)) = \text{sign}(\sigma)f(T).$$

Using relations of the kind found in condition (iii) we obtain the following

$$\left\{\begin{array}{r} \displaystyle\sum_{\substack{i_1,\ldots,i_t\in I\\ i_1<\ldots<i_t}} f(Y^s(i_1,\ldots,i_t)) - \sum_{o_1\in O_h}\sum_{\substack{i_1,\ldots,i_{t-1}\in I\\ i_1<\ldots<i_{t-1}}} f(Y^s(o_1,i_1,\ldots,i_{t-1})) \\ \displaystyle + \sum_{o_1,o_2\in O_h}\sum_{\substack{i_1,\ldots,i_{t-2}\in I\\ i_1<\ldots<i_{t-2}}} f(Y^s(o_1,o_2,i_1,\ldots,i_{t-2})) \\ \displaystyle -\ldots+(-1)^{t-1}\sum_{o_1,o_2,\ldots,o_{t-1}\in O_h}\sum_{i_1\in I} f(Y^s(o_1,o_2,\ldots,o_{t-1},i_1)) \end{array}\right\} = 0. \qquad (*)$$

Since all but the first sum in this equation involve elements $f(T')$ where T' contains at least one entry from O_h in column s, the element

$$f(Y^s(r_1,\ldots,r_s)) = \text{sign}(\sigma)f(T)$$

occurs exactly once in the whole expression. By taking $f(Y^s(r_1,\ldots,r_s))$ over to the right hand side of the equation we obtain an expression for $f(T)$ as a linear combination of elements $f(T')$ for various T'.

We are interested in finding which $f(T')$ have non-zero coefficients in that expression. Let γ be a permutation of $\{1,\ldots,t\}$. Then γ determines a permutation of $Y^s(j_1,\ldots,j_t)$ sending it to $Y^s(j_{\gamma(1)},\ldots,j_{\gamma(t)})$. As a permutation of the tableau this is even, since entries are permuted in pairs. Hence

$$f(Y^s(j_1,\ldots,j_t)) = f(Y^s(j_{\gamma(1)},\ldots,j_{\gamma(t)}))$$

by condition (ii). So we can write $Y^s\{j_1,\ldots,j_t\}$ instead of $Y^s(j_1,\ldots,j_t)$ as the order doesn't matter.

Let $j_1,\ldots,j_t \in I$ be distinct with $\{j_1,\ldots,j_t\}\cap O_h = \{p_1,\ldots,p_\alpha\} \neq \emptyset$, and let $T' = Y^s\{j_1,\ldots,j_t\}$. By the above we may assume that $T' = Y^s\{j_1',\ldots,j_{t-\alpha}',p_1,\ldots,p_\alpha\}$, where $\{j_1',\ldots,j_{t-\alpha}'\} = \{j_1,\ldots,j_t\}\setminus\{p_1,\ldots,p_\alpha\}$. We wish to show that the coefficient of $f(T')$ on the left hand side of equation $(*)$ is zero.

Let $\beta\in\{0,\ldots,\alpha\}$. Then $f(T') = f(Y^s\{j_1',\ldots,j_{t-\alpha}',p_1,\ldots,p_\alpha\})$ occurs with coefficient $(-1)^\beta\binom{\alpha}{\beta}$ in

$$\sum_{o_1,\ldots,o_\beta\in O_h}\ \sum_{\substack{i_1,\ldots,i_{t-\beta}\in I\\ i_1<\ldots<i_{t-\beta}}} k_{Y^s\{o_1,\ldots,o_\beta,i_1,\ldots,i_{t-\beta}\}}.$$

Hence the coefficient of $f(T')$ in the left hand side of equation $(*)$ is

$$1-\alpha+\binom{\alpha}{2}-\ldots+(-1)^\alpha\binom{\alpha}{\alpha} = \sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i} = 0.$$

So if $\{j_1,\ldots,j_t\} \cap O_h \neq \emptyset$ then $f(Y^s\{j_1,\ldots,j_t\})$ has coefficient 0 in (*).

Suppose $T' \in \mathcal{T}$ satisfies $T \neq T'$, $f(T') \neq 0$ and that the coefficient of $f(T')$ in equation (*) is non-zero. Then $T' = Y^s(j_1,\ldots,j_t)$ where $\{j_1,\ldots,j_t\} \cap O_h = \emptyset$, $\{j_1,\ldots,j_t\} \neq \{r_1,\ldots,r_t\}$ and $\{j_1,\ldots,j_t\} \cap (\{t_{i_1},\ldots,t_{i_h}\} \setminus \{r_1,\ldots,r_t\}) = \emptyset$ (otherwise $f(Y^s\{j_1,\ldots,j_t\}) = 0$ by condition (i)). So $\{j_1,\ldots,j_t\}$ must contain at least one element not in $\{t_{i_1},\ldots,t_{i_h}\}$. Any element in $\{j_1,\ldots,j_t\}$ not in $\{t_{i_1},\ldots,t_{i_h}\}$ must be greater than $\overline{h-1}$. So T' is obtained from $\sigma(T)$ by replacing some entries by ones of greater modulus. Thus we have the required expression.

□

5.3.15 Proposition. *The set $\{\overline{v}_T\,;\ T$ is symplectic$\}$ spans $V^{sp}_{\lambda,\mathbf{k}}$.*

Proof. Since $V^{sp}_{\lambda,\mathbf{k}} \subset V^{gl}_{\lambda,\mathbf{k}}$ all elements $v \in V^{sp}_{\lambda,\mathbf{k}}$ satisfy the relations of Definition 2.3.1. Let v be any non-zero element of $V^{sp}_{\lambda,\mathbf{k}}$. Then v can be expressed as

$$v = \sum_{T \in \mathcal{T}} k_T t_T \quad k_T \in \mathbf{k}$$

where $t_T = v_{t_1} \otimes \ldots \otimes v_{t_r}$ and $t_1,\ldots,t_r$ are the entries in positions $1,\ldots,r$ of T.

The relations in Definition 2.3.1 give relations on the coefficients k_T as follows:-

(i) $k_T = 0$ if T has equal entries in two distinct squares belonging to the same column;

(ii) whenever $T, T' \in \mathcal{T}$ satisfy that T' is obtained from T by interchanging two squares of the same column

$$k_T + k_{T'} = 0;$$

(iii) for all $h \in \{1,\ldots,\lambda_1 - 1\}$ and all subsets $J_h \subset I_h$ and $J_{h+1} \subset I_{h+1}$ such that $|J_h| + |J_{h+1}| > |I_h|$ we have

$$\sum_{\sigma \in S(J)} \operatorname{sign}(\sigma) k_{\sigma(T)} = 0$$

where σ ranges over the set $S(J)$ of permutations of $\{1,\ldots,r\}$ which are the identity outside $J_h \cup J_{h+1}$ and such that $\sigma(i) < \sigma(j)$ for $i < j$ in J_h and for $i < j$ in J_{h+1}.

During this proof we say that two λ-tableaux T and T' are equivalent if the sum of the moduli of the entries in the i^{th} column of T is equal to the sum of the moduli of the entries in the i^{th} column of T' for all $i \in \{1,\ldots,\lambda_1\}$. Let (T) denote the equivalence class of T. Define the ordering $T \preceq T'$ if the sum of the moduli of the entries in the last i columns of T is less than or equal to the sum of the moduli of the entries in the last i columns of T' for all $i \in \{1,\ldots,\lambda_1\}$. Clearly $T_1 \preceq T_2$ and $T_2 \preceq T_1$ implies that $(T_1) = (T_2)$. Therefore this gives a partial ordering on the set of equivalence classes of tableaux in $\mathcal{T}$.

Choose $T^m \in \mathcal{T}$ with $k_{T^m} \neq 0$ in the expression

$$v = \sum_{T \in \mathcal{T}} k_T t_T$$

but $k_T = 0$ for all $T \in \mathcal{T}$ such that $(T^m) \prec (T)$. We know the entries within each column of T^m are distinct; otherwise, by condition (i), $k_{T^m} = 0$. Using condition (ii), we can assume that T^m is strictly increasing down its columns.

In fact, we claim that T^m is semistandard. Suppose not. As T^m is strictly increasing down its columns, there must be a row of T^m which is not non-decreasing. Let this be row $s \in \{1, \ldots, \mu_1\}$. So there are two columns T^m_h and T^m_{h+1} of the form

i_1	j_1
$\vdots$	$\vdots$
i_s	j_s
$\vdots$	$\vdots$
i_{μ_h}	$j_{\mu_{h+1}}$

with $i_s > j_s$. Hence $j_1 < \ldots < j_s < i_s < i_{s+1} < \ldots < i_{\mu_h}$. Let I_h and I_{h+1} denote the h^{th} and $(h+1)^{\text{th}}$ columns of the the leading λ-tableau respectively. Let $J_h \subset I_h$ be the set of squares containing $i_s, \ldots, i_{\mu_h}$ and let $J_{h+1} \subset I_{h+1}$ be the set of squares containing $j_1, \ldots, j_s$. The coefficients k_T satisfy the Garnir relation

$$\sum_{\sigma \in S(J)} \text{sign}(\sigma) k_{\sigma(T^m)} = 0$$

where σ ranges over the set $S(J)$ of all permutations of $\{1, \ldots, r\}$ which are the identity outside $J_h \cup J_{h+1}$ and such that $\sigma(i) < \sigma(j)$ for $i < j$ in J_h and for $i < j$ in J_{h+1}.

Hence there exists $\sigma \in S(J)$ with $\sigma \neq 1$ and $k_{\sigma(T^m)} \neq 0$. As $\sigma \neq 1$ it must permute some of the entries of J_h into J_{h+1} and *vice versa.* Hence $(T^m) \prec (\sigma(T^m))$, contradicting the maximality of T^m such that $k_{T^m} \neq 0$. Consequently T^m is semistandard.

We also claim that T^m is symplectic. Assume not. Let $f : \mathcal{T} \to \mathbf{k}$ be the function taking T to k_T, the coefficient of t_T in v. By properties (i) and (ii) and Proposition 5.3.12and the discussion following that proposition we see that this function satisfies the conditions of Lemma 5.3.14. Therefore k_{T^m} can be expressed as a linear combination of numbers $k_{T'}$ where each $T' \in \mathcal{T}$ satisfies $(T^m) \prec (T')$. Therefore there is a $T' \in \mathcal{T}$ such that $(T^m) \prec (T')$ and $k_{T'} \neq 0$. This again contradicts the maximality of T^m, and so T^m is symplectic.

Let

$$v' = v - k_{T^m} \overline{v}_{T^m}.$$

Since $\overline{v}_{T^m} \in V^{sp}_{\lambda,\mathbf{k}}$ we have $v' \in V^{sp}_{\lambda,\mathbf{k}}$. Consider the nature of $\overline{v}_{T^m}$. We have

$$\overline{v}_{T^m} = \overline{\psi}_{T^m} + \sum_{T'} \pm \overline{\psi}_{T'}$$

where the sum is over various semistandard λ-tableaux T' possibly with multiplicity. By the proofs of Lemmas 5.2.9, 5.2.11and 5.2.13 we see that each T' satisfies $(T') \prec (T^m)$.

Let T be a semistandard λ-tableau. Then

$$\overline{\psi}_T = \sum_{T+} \overline{\phi}_{T+}$$

where the sum is over all row permutations T^+ of T. Since T is non-decreasing along rows, we have $(T^+) \prec (T)$ whenever $T^+ \neq T$.

Consequently

$$\overline{v}_{T^m} = \overline{\phi}_{T^m} + (\text{ sum of terms } \overline{\phi}_{T'})$$

where each T' satisfies $(T') \prec (T^m)$. However $\overline{\phi}_{T^m} = \sum_{\sigma \in C(\lambda)} \operatorname{sign}(\sigma)\sigma\overline{t}_{T^m}$ and for any $\sigma \in C(\lambda)$ then $(T^m) = (\sigma(T^m))$. Since T^m is symplectic, it is strictly increasing down columns, and $\sigma(T^m)$ is not symplectic when $\sigma \neq 1$, $\sigma \in C(\lambda)$.

Similarly, if $(T') \prec (T^m)$ then $\overline{\phi}_{T'} = \sum \pm \overline{t}_{T''}$ where each T'' satisfies $(T'') \prec (T^m)$. So we have

$$\overline{v}_{T^m} = \overline{t}_{T^m} + (\text{ sum of terms } \overline{t}_{T^1}) + (\text{ sum of terms } \overline{t}_{T^2})$$

where each T^1 is not symplectic and satisfies $T^1 \in (T^m)$ and where each T^2 satisfies $(T^2) \prec (T^m)$.

Let

$$v' = \sum_{T' \in T} k'_{T'}\overline{t}_{T'}$$

for some $k'_{T'} \in \mathbf{k}$.

Let $T^{m\prime}$ satisfy $k'_{T^{m\prime}} \neq 0$ and whenever $(T^{m\prime}) \prec (T')$ then $k'_{T'} = 0$. By the previous argument we know $T^{m\prime}$ can be chosen to be symplectic. Since

$$v' = v - k_{T^m}\overline{v}_{T^m}$$

we have that $(T^{m\prime}) \preceq (T^m)$. Suppose $(T^{m\prime}) = (T^m)$. Then the number of λ-tableaux $T' \in (T^{m\prime})$ such that T' is symplectic and satisfies $k'_{T'} \neq 0$ is one less than the number of λ-tableaux $T \in (T^m)$ such that T is symplectic and satisfies $k_T \neq 0$.

We repeat the process by defining

$$v'' = v' - k'_{T^{m\prime}}\overline{v}_{T^{m\prime}}.$$

Since the number of λ-tableaux is finite, by repeating this process enough times, we obtain an expression for v as a linear combination of terms $\overline{v}_{T'}$ with each T' symplectic.

□

5.3.16 Theorem. *The set $\{\overline{v}_T; T \in T$ is symplectic$\}$ form a basis of $V^{sp}_{\lambda,\mathbf{k}}$.*

Proof. By Proposition 5.2.14we know that this set is linearly independent over $\mathbf{k}$, and by Proposition 5.3.15we know that it spans $V^{sp}_{\lambda,\mathbf{k}}$.

□

5.3.17 Corollary.

$$V^{sp}_{\lambda,\mathbf{k}} \cong \mathbf{k} \otimes_{\mathbf{Z}} V^{sp}_{\lambda,\mathbf{Z}}$$

Proof. Everything we have done in showing that the set of $\overline{v}_T$ where T ranges over all symplectic λ-tableau is a basis is valid in $V^{sp}_{\lambda,\mathbf{Z}}$. Hence the set $\{v_T; T \in \mathcal{T}$ is symplectic$\}$ forms a $\mathbf{Z}$-basis for $V^{sp}_{\lambda,\mathbf{Z}}$. The result follows.

□

5.3.18 Lemma. *λ is the highest weight of $V^{sp}_{\lambda,\mathbf{k}}$.*

Proof. Let $v \in V^{sp}_{\lambda,\mathbf{k}}$. Then v has the form $v = u\phi_\lambda$ for some $u \in \mathcal{U}^-_{sp,\mathbf{k}}$. Hence V can be obtained by the action of a product of elements in $\mathfrak{n}^- \subset \mathfrak{sp}_{2n}(\mathbf{k})$, where $\mathfrak{n}^-$ is the sum of all the negative root spaces in $\mathfrak{sp}_{2n}(\mathbf{k})$. Under the action of each element of $\mathfrak{n}^-$ the weight decreases. So λ is strictly greater than the weights of all vectors in $V^{sp}_{\lambda,\mathbf{k}} \setminus \{\mathbf{k}\phi_\lambda\}$.

□

5.3.19 Lemma. *$V^{sp}_{\lambda,\mathbf{k}}$ has a unique maximal $Sp_{2n}(\mathbf{k})$-submodule $M^{sp}_{\lambda,\mathbf{k}}$ and the factor module $F^{sp}_{\lambda,\mathbf{k}}$, given by*

$$F^{sp}_{\lambda,\mathbf{k}} = V^{sp}_{\lambda,\mathbf{k}}/M^{sp}_{\lambda,\mathbf{k}},$$

is an irreducible polynomial representation of highest weight λ.

Proof. Let M be any proper $Sp_{2n}(\mathbf{k})$-submodule of $V^{sp}_{\lambda,\mathbf{k}}$. Since ϕ_λ generates the whole module we must have $k\phi_\lambda \notin M$ for all $k \in \mathbf{k}$. Hence

$$M \subset \bigoplus_{\substack{\mu \in X(T) \\ \mu < \lambda}} \left(V^{sp}_{\lambda,\mathbf{k}}\right)^\mu.$$

Let $M^{sp}_{\lambda,\mathbf{k}}$ be the sum of all proper $Sp_{2n}(\mathbf{k})$-submodules of $V^{sp}_{\lambda,\mathbf{k}}$. Then

$$M^{sp}_{\lambda,\mathbf{k}} \subset \bigoplus_{\substack{\mu \in X(T) \\ \mu < \lambda}} \left(V^{sp}_{\lambda,\mathbf{k}}\right)^\mu,$$

and $k\phi_\lambda \notin M^{sp}_{\lambda,\mathbf{k}}$ for all $k \in \mathbf{k}$. So $M^{sp}_{\lambda,\mathbf{k}}$ is a proper submodule of $V^{sp}_{\lambda,\mathbf{k}}$. It is therefore the unique maximal $Sp_{2n}(\mathbf{k})$-submodule of $V^{sp}_{\lambda,\mathbf{k}}$. So $F^{sp}_{\lambda,\mathbf{k}}$ is an irreducible $Sp_{2n}(\mathbf{k})$-module and has highest weight λ.

□

6

Schur Modules.

This chapter is an extension to the symplectic group of the work on Schur modules for the general linear group described in Green [1], 50-64. Henceforth, we refer to the $Gl_{2n}(\mathbf{k})$-module $D_{\lambda,\mathbf{k}}$ as $D^{gl}_{\lambda,\mathbf{k}}$ for clarity. let V be a $2n$-dimensional vector space over $\mathbf{k}$. Then V is a module for $GL_{2n}(\mathbf{k})$ and $Sp_{2n}(\mathbf{k})$ in the natural way, and V will have a symplectic basis which we denote by

$$\{v_1, v_2, \ldots, v_n, v_{\overline{n}}, v_{\overline{n-1}}, \ldots, v_{\overline{1}}\}.$$

Let $I = \{1, \ldots, n\}$ and $\overline{I} = \{\overline{1}, \ldots, \overline{n}\}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $\lambda_1 + \ldots + \lambda_n = r$. In this chapter we assume all λ-tableaux have entries from $I \cup \overline{I}$.

6.1 The Induced Module.

For $i, j \in I \cup \overline{I}$ recall from Chapter 3 the coefficient functions

$$c_{i,j} : GL_{2n}(\mathbf{k}) \to \mathbf{k}$$
$$d_{i,j} : Sp_{2n}(\mathbf{k}) \to \mathbf{k}$$

where $d_{i,j}$ is the restriction of $c_{i,j}$ to the symplectic group. The algebra of polynomials in the $c_{i,j}$ is called $A_{\mathbf{k}}(\overline{n})$, and recall that there is an ideal $\mathfrak{I} \subset A_{\mathbf{k}}(\overline{n})$ generated by certain expressions in the $c_{i,j}$ which vanish when restricted to $Sp_{2n}(\mathbf{k})$. We denote the algebra of polynomials in the $d_{i,j}$ by $A^{sp}_{\mathbf{k}}(\overline{n})$, and this satisfies $A_{\mathbf{k}}(\overline{n})/\sqrt{\mathfrak{I}} = A^{sp}_{\mathbf{k}}(\overline{n})$. Let $I(\overline{n}, r)$ denote the set of r-tuples with entries from $I \cup \overline{I}$.

The algebra $A^{sp}_{\mathbf{k}}(\bar{n})$ forms a left Sp-module under the action given by

$$s \circ f(g) = f(gs)$$

for all $s, g \in Sp$ and all $f \in A^{sp}_{\mathbf{k}}(\bar{n})$. It also forms a right Sp-module under the action

$$f \circ s(g) = f(sg)$$

for all $s, g \in Sp$ and all $f \in A^{sp}_{\mathbf{k}}(\bar{n})$. Under these actions $A^{sp}_{\mathbf{k}}(\bar{n})$ is a polynomial module since matrix multiplication is a polynomial function in the coefficients of the matrices.

Furthermore, if $A^{sp}_{\mathbf{k}}(\bar{n},r) \subset A^{sp}_{\mathbf{k}}(\bar{n})$ is the subset of polynomials in the $d_{i,j}$ which are homogeneous of degree r, then $A^{sp}_{\mathbf{k}}(\bar{n},r)$ is a left and right Sp-submodule of $A^{sp}_{\mathbf{k}}(\bar{n})$.

We are interested in left Sp-submodules inside the space $A^{sp}_{\mathbf{k}}(\bar{n})$. However, the right Sp-action given above is useful in obtaining a left Sp-module as follows.

6.1.1 Definition. Let $I^{sp}_{\lambda,\mathbf{k}} \subset A^{sp}_{\mathbf{k}}(\bar{n})$ be defined by

$$I^{sp}_{\lambda,\mathbf{k}} = \{f \in A^{sp}_{\mathbf{k}}(\bar{n}); f(bg) = \lambda(b)f(g) \text{ for all } b \in B^-,\ g \in Sp\}$$

Here λ denotes the lift of $\lambda : T \to \mathbf{k}$ to B^- by putting U^-_{sp} in the kernel. Let $\mathbf{k}(\lambda)$ denote the right B^--module affording λ. Then $I_{\lambda,\mathbf{k}} = \mathrm{Ind}^{Sp}_{B^-}\mathbf{k}(\lambda)$.

The space $I^{sp}_{\lambda,\mathbf{k}}$ is stable under the left action of Sp, and forms a polynomial Sp-module. Its dimension is given by Weyl's dimension formula, which is equal to the number of symplectic λ-tableaux.

6.2 The Module $D^{Sp}_{\lambda,\mathbf{k}}$.

6.2.1 Definition.

For any λ-tableau T let the bideterminant $D_T \in A^{sp}(\overline{n},r)$ be the restriction to the symplectic group of the bideterminant $b_T \in A_{\mathbf{k}}(\overline{n},r)$.

Thus when T is a (1^m)-tableau of the form $T = \begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline \vdots \\ \hline t_m \\ \hline \end{array}$ for some $m \in \mathbb{N}$ the bideterminant $D_T \in A^{sp}_{\mathbf{k}}(\overline{n})$ is given by

$$D_T = \begin{vmatrix} d_{1,t_1} & \dots & d_{1,t_m} \\ \vdots & & \vdots \\ d_{m,t_1} & \cdots & d_{m,t_m} \end{vmatrix} = \sum_{\sigma \in S(m)} \mathrm{sign}(\sigma) d_{1,t_{\sigma(1)}} \dots d_{m,t_{\sigma(m)}},$$

and in general, when T is a λ-tableau, and $T_1, \dots, T_{\lambda_1}$ are the columns of T, we have

$$D_T = D_{T_1} \dots D_{T_{\lambda_1}}.$$

Let $\pi : A_{\mathbf{k}}(\overline{n}) \to A^{sp}_{\mathbf{k}}(\overline{n})$ be the canonical map so that $\pi(b_T) = D_T$ for any λ-tableau T.

6.2.2 Definition. Define $D^{sp}_{\lambda,\mathbf{k}}$ to be the $\mathbf{k}$-span of the bideterminants D_T as T ranges over all λ-tableaux.

So $D^{sp}_{\lambda,\mathbf{k}}$ is the restriction to the symplectic group of $D^{gl}_{\lambda,\mathbf{k}}$, that is $\pi(D^{gl}_{\lambda,\mathbf{k}}) = D^{sp}_{\lambda,\mathbf{k}}$. In fact, we have the following commutative diagram

$$\begin{array}{ccc} A_{\mathbf{k}}(\overline{n}) & \xrightarrow{Sp_{2n}(\mathbf{k})} & A_{\mathbf{k}}(\overline{n}) \\ \downarrow \pi & & \downarrow \pi \\ A^{sp}_{\mathbf{k}}(\overline{n}) & \xrightarrow{Sp_{2n}(\mathbf{k})} & A^{sp}_{\mathbf{k}}(\overline{n}) \end{array}$$

Hence $D^{sp}_{\lambda,\mathbf{k}}$ is an $Sp_{2n}(\mathbf{k})$-submodule of $A^{sp}_{\mathbf{k}}(\overline{n})$.

Since the set of bideterminants b_T, as T ranges over all semistandard λ-tableaux, forms a basis for $D^{gl}_{\lambda,\mathbf{k}}$ we know that the set of bideterminants D_T, as T ranges over all semistandard λ-tableaux, spans $D^{sp}_{\lambda,\mathbf{k}}$.

We wish to show that the set of bideterminants D_T, as T ranges over all symplectic λ-tableaux, forms a basis for $D^{sp}_{\lambda,\mathbf{k}}$. We begin by showing that they form a spanning set for $D^{sp}_{\lambda,\mathbf{k}}$. It will be sufficient to show that a bideterminant D_T, of a semistandard λ-tableau T, can be expressed as a linear combination of bideterminants $D_{T'}$, of symplectic λ-tableaux T'. When the characteristic of $\mathbf{k}$ is zero or sufficiently large, the quadratic relations defined in Chapter 3 will give us enough information to do this.

6.3 The Spanning Property when Char(k) is Zero or Large.

In the case when the characteristic of $\mathbf{k}$ is zero or sufficiently large the simple expansion operators will provide enough information to show that the bideterminants of the symplectic λ-tableaux span $D^{sp}_{\lambda,\mathbf{k}}$. We aim to show this, and begin by studying the simple relations between bideterminants.

Recall from Chapter 3 that the quadratic form $Q_{i,j}$, where $i,j \in I \cup \overline{I}$, is given by

$$Q_{i,j} = \sum_{k=1}^{n} \begin{vmatrix} d_{i,k} & d_{i,\overline{k}} \\ d_{j,k} & d_{j,\overline{k}} \end{vmatrix}.$$

The quadratic relations are

$$\begin{aligned} Q_{i,j} &= 0 \\ Q_{i,\overline{j}} &= 0 \\ Q_{\overline{i},j} &= 0 \\ Q_{\overline{i},\overline{j}} &= 0 \\ Q_{k,\overline{k}} &= 1 \end{aligned}$$

for all $i,j \in I$ such that $i < j$ and all $k \in I$. These relations define the symplectic group.

The quadratic relations of the form $Q_{i,j} = 0$, for $i,j \in I$ such that $i < j$, give relations between the bideterminants D_T. These relations are obtained as follows.

Let $m \leq n-2$ and let T be the (1^{m+2})-tableau given by

$$T = \begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline \vdots \\ \hline t_m \\ \hline i \\ \hline \bar{i} \\ \hline \end{array}\,.$$

Then $D_T = \det A$ where

$$A = \begin{pmatrix} d_{1,t_1} & \dots & d_{1,t_m} & d_{1,i} & d_{1,\bar{i}} \\ d_{2,t_1} & \dots & d_{2,t_m} & d_{2,i} & d_{2,\bar{i}} \\ \vdots & & \vdots & \vdots & \vdots \\ d_{m+2,t_1} & \cdots & d_{m+2,t_m} & d_{m+2,i} & d_{m+2,\bar{i}} \end{pmatrix}.$$

By applying the Laplace expansion of determinants we get

$$D_T = \sum_{A'} \epsilon(A') \det A' \det A''$$

where A' runs over all m^{th} order matrices obtained from the first m columns of A and any choice of rows, where A'' is the 2×2 matrix obtained by omitting these rows and columns and where $\epsilon(A') = (-1)^{\nu}$ with ν equal to the number of transpositions required to bring the rows of A' to the first m positions. We rewrite this as

$$D_T = \sum_{\substack{a_1,\dots,a_m \\ a_1<\dots<a_m}} \operatorname{sign}(\sigma) \det A' \det A''$$

where $\sigma = \begin{pmatrix} 1 & \dots & m & m+1 & m+2 \\ a_1 & \dots & a_m & b_1 & b_2 \end{pmatrix}$ such that $\{b_1, b_2\} = \{1, \dots, m+2\} \setminus \{a_1, \dots, a_m\}$ with $b_1 < b_2$ and

$$A' = \begin{pmatrix} d_{a_1,t_1} & \dots & d_{a_1,t_m} \\ \vdots & & \vdots \\ d_{a_m,t_1} & \cdots & d_{a_m,t_m} \end{pmatrix} \quad \text{and} \quad A'' = \begin{pmatrix} d_{b_1,i} & d_{b_1,\bar{i}} \\ d_{b_2,i} & d_{b_2,\bar{i}} \end{pmatrix}.$$

If we denote a 2×2 matrix of the form of A'' by $D^{b_1,b_2}_{i,\bar{i}}$ then we can write

$$D_T = \sum_{\substack{a_1,\dots,a_m \\ a_1<\dots<a_m}} \operatorname{sign}(\sigma) \det A' \det \left(D^{b_1,b_2}_{i,\bar{i}}\right).$$

For any $b_1, b_2 \in \{1, \ldots, n\}$ the quadratic relations give us $\sum_{i=1}^{n} \det\left(D_{i,\bar{i}}^{b_1,b_2}\right) = 0$. Hence the following expression must be zero.

$$\sum_{\substack{a_1,\ldots,a_m \\ a_1<\ldots<a_m}} \operatorname{sign}(\sigma) \det A' \sum_{i=1}^{n} \left(\det D_{i,\bar{i}}^{b_1,b_2}\right).$$

Reordering this gives the equation

$$\sum_{i=1}^{n} \sum_{\substack{a_1,\ldots,a_m \\ a_1<\ldots<a_m}} \operatorname{sign}(\sigma) \det A' \left(\det D_{i,\bar{i}}^{b_1,b_2}\right) = 0,$$

which is equivalent to

$$\sum_{i=1}^{n} D_{\begin{array}{|c|}\hline t_1\\\hline t_2\\\hline \vdots\\\hline t_m\\\hline i\\\hline \bar{i}\\\hline\end{array}} = D_{\begin{array}{|c|}\hline t_1\\\hline t_2\\\hline \vdots\\\hline t_m\\\hline 1\\\hline \bar{1}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline t_1\\\hline t_2\\\hline \vdots\\\hline t_m\\\hline 2\\\hline \bar{2}\\\hline\end{array}} + \ldots + D_{\begin{array}{|c|}\hline t_1\\\hline t_2\\\hline \vdots\\\hline t_m\\\hline n\\\hline \bar{n}\\\hline\end{array}} = 0.$$

More generally, let $i, j \in I$ and let $m \in \mathbb{N}$ such that $m + 2 \leq n$. Let $t = (t_1, \ldots, t_m)$ be an m-tuple with entries from $I \cup \bar{I}$ satisfying $t_1 < \ldots < t_m$. For any $k \in I$ we can construct a (1^{m+2})-tableau T_k by putting a k in the i^{th} row, a $\bar{k}$ in the j^{th} row, and filling the remaining squares in T_k with $t_1, \ldots, t_m$ in increasing order down the column. Then the quadratic relations give the following

$$\sum_{k=1}^{n} D_{T_k} = 0.$$

Such an equation is called a *simple relation* between bideterminants.

6.3.1 Example. Let $i = 1$, $j = 2$, $m = 2$, $n = 5$ and $t = (1, 3)$. Then

$$\sum_{k=1}^{n} D_{T_k} = D_{\begin{array}{|c|}\hline 1\\\hline \bar{1}\\\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \bar{3}\\\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 4\\\hline \bar{4}\\\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 5\\\hline \bar{5}\\\hline 1\\\hline 3\\\hline\end{array}} = 0.$$

Since $D_T = 0$ whenever two different squares in the same column of T contain equal entries, this is equivalent to

$$D_{\begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 4\\\hline \bar{4}\\\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 5\\\hline \bar{5}\\\hline 1\\\hline 3\\\hline\end{array}} = 0,$$

which, by other properties of determinants, is equivalent to

$$D_{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \bar{2} \\ \hline 3 \\ \hline \end{array}} + D_{\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \bar{4} \\ \hline \end{array}} + D_{\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 5 \\ \hline \bar{5} \\ \hline \end{array}} = 0.$$

As can be seen in the above example, moving the entries into increasing order doesn't affect the relation. This is because once k and $\bar{k}$ are adjacent, moving them to another two adjacent squares produces an even permutation, and so the sign of the permutation to move them to adjacent places, and reorder them among the remaining entries, depends only on i and j.

6.3.2 Definition. Let $m \in \mathbb{N}$ satisfy $m \leq n-2$. Let the *simple expansion operator* $E_1 : D^{sp}_{(1^m),\mathbf{k}} \to D^{sp}_{(1^{m+2}),\mathbf{k}}$ be the linear extension of

$$E_1 : D_{\begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline \vdots \\ \hline t_m \\ \hline \end{array}} \longmapsto D_{\begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline \vdots \\ \hline t_m \\ \hline 1 \\ \hline \bar{1} \\ \hline \end{array}} + D_{\begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline \vdots \\ \hline t_m \\ \hline 2 \\ \hline \bar{2} \\ \hline \end{array}} + \ldots + D_{\begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline \vdots \\ \hline t_m \\ \hline n \\ \hline \bar{n} \\ \hline \end{array}}.$$

By the preceding discussion we have shown that for all $m \leq n-2$ the module $D^{sp}_{(1^m),\mathbf{k}}$ is in the kernel of E_1. Let Y be the (1^2)-tableau given by $Y = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}$. Then in the example above, we obtained the equation $E_1(D_Y) = 0$.

6.4 The Case when λ is a Column.

We wish to show that given any bideterminant D_T of a semistandard λ-tableau T we can express D_T as a linear combination of bideterminants $D_{T'}$ where the T' are symplectic λ-tableaux. Since we already know from Green [1] that the bideterminants of all the semistandard λ-tableaux span $D^{sp}_{\lambda,\mathbf{k}}$ that will be sufficient to show that the bideterminants of all the symplectic λ-tableaux span $D^{sp}_{\lambda,\mathbf{k}}$. At first we intend to do this using only the simple relations between bideterminants, assuming that the characteristic of the field is zero or sufficiently large. We begin with the simplest case, that is, when λ is a column.

Until further notice let $m \in \mathbb{N}$ with $m \leq n$ and let $\lambda = (1^m)$.

6.4.1 Definitions. Let T be a λ-tableau with entries from $I \cup \bar{I}$. Then T can be described by its *repeat set* R and its *single set* S, where

$$R = \{r \in I; r \text{ and } \bar{r} \text{ both occur in } T\}$$

and S contains the remaining entries of T. Conversely, given sets $R \subset I$ and $S \subset I \cup \bar{I}$ such that $\{r, \bar{r} \,;\, r \in R\} \cap S = \emptyset$ and $2|R| + |S| = m$, there is a unique semistandard λ-tableau described by R and S. This tableau is written $T_{R,S}$, or T_R for short when $S = \emptyset$.

For example, let T be the (1^5)-tableau

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \bar{3} \\ \hline \bar{5} \\ \hline 6 \\ \hline \bar{6} \\ \hline \end{array}$$

Then $R = \{3, 6\}$ and $S = \{1, \bar{5}\}$.

Let Y be a (1^{m-2})-tableau. Then the simple expansion $E_1(Y)$ is a linear combination of bideterminants of (1^m)-tableaux. We define the *relation set* P of $E_1(Y)$ to be the repeat set of the tableau Y, and we define the *single set* S of $E_1(Y)$ to be the single set of Y. Given two sets $P \subseteq I$ and $S \subseteq I \cup \bar{I}$ with $\{p, \bar{p} \,;\, p \in P\} \cap S = \emptyset$ and $2|P| + |S| = m - 2$ there is a unique simple expansion $E_{P,S} = E_1(T_{P,S})$ with repeat set P and single set S. In the case $S = \emptyset$ we denote the expression $E_{P,S}$ by E_P.

If the λ-tableau T_R is symplectic then its repeat set R is said to be symplectic. Clearly, $R \subseteq I$ is symplectic if and only if R has the form $R = \{r_1, \ldots, r_\rho\}$ with $r_1 < \ldots < r_\rho$ and $r_j \geq 2j$ for all $j = 1, \ldots, \rho$.

A simple relation $E_{P,S} = 0$ only involves bideterminants of λ-tableaux with single sets equal to S. So we fix a single set $S \subset I \cup \bar{I}$ and consider the information about relations between bideterminants given by the simple relations with single sets S. The simplest case is $S = \emptyset$, and we begin with that.

Tableaux in which All Entries are Repeated.

Assume $\lambda = (1^m)$, m is even and let $\rho = \frac{m}{2}$. If the single set of a λ-tableau T is empty then the repeat set R is given by $R = \{r_1, \ldots, r_\rho\}$ and the set of entries of T is $\{r_1, \overline{r_1}, \ldots, r_\rho, \overline{r_\rho}\}$. In this case we say all entries of T are repeated.

6.4.2 Definitions. Let X be the set of all semistandard non-symplectic λ-tableaux with empty single set. Then X is of the form

$$X = \{T_R \,;\, R \subset I \text{ with } |R| = \rho \text{ and } R \text{ is not symplectic}\}.$$

Let E be the subset of all simple expansions given by

$$E = \{E_P \,;\, P \subset I \text{ with } |P| = \rho - 1\}.$$

Then E is the set of all simple expansions E_P which involve one or more of the bideterminants of the tableaux in X.

X is a set of (1^m)-tableaux, and E is a set of linear combinations of bideterminants of (1^m)-tableaux. Our aim is to show that the relations $E_P = 0$ for all $E_P \in E$ are sufficient to express all bideterminants of tableaux in X as linear combinations of bideterminants of symplectic (1^m)-tableaux, when the characteristic of the base field is zero or sufficiently large.

In order to partition the sets X and E in a useful way we need the following definitions.

6.4.3 Definitions. A word is a non-empty finite sequence of numbers, with or without hats, of the form

$$(a_n, a_{n-1}, \ldots, a_l) \qquad \text{with } a_j \in \{j, \hat{j}\}$$

for all $j \in \{l, \ldots, n\}$ and for some $l \in \{1, \ldots, n\}$.

For example, when $n = 7$ then $(7, \hat{6}, \hat{5}, 4, \hat{3})$ and $(\hat{7}, 6, 5)$ are words, and the total number of words is $2^n + 2^{(n-1)} + \ldots + 2$.

Let ω be a word. Then ω is defined to be of *type 1* if it has $\rho - 1$ unhatted elements, and to be of *type 2* if it contains $n - 2\rho + 1$ more hatted than unhatted elements. So if $n = 7$ and $m = 6$ then $\rho = 3$ and $(7, \hat{6}, 5)$ is of type 1, and $(\hat{7}, 6, \hat{5}, \hat{4})$ is of type 2.

A *complete word* is a word of type 1 or type 2 which has no proper initial segment of type 1 or type 2. It is possible for a word to be of type 1 and type 2, but a complete word cannot have this property. We denote the set of complete words by Ω.

6.4.4 Example. When $n = 7$ and $\rho = 3$ then $|\Omega| = 8$ and the following are all the complete words.

$$\begin{array}{llll} \omega_1 = (\hat{7}, \hat{6}) & \omega_2 = (\hat{7}, 6, \hat{5}, \hat{4}) & \omega_3 = (\hat{7}, 6, \hat{5}, 4) & \omega_4 = (\hat{7}, 6, 5) \\ \omega_5 = (7, \hat{6}, \hat{5}, \hat{4}) & \omega_6 = (7, \hat{6}, \hat{5}, 4) & \omega_7 = (7, \hat{6}, 5) & \omega_8 = (7, 6) \end{array}$$

We define a total ordering on the set of complete words as follows. Let ω and ω' be two distinct words. Then $\omega < \omega'$ if and only if the largest element on which ω and ω' disagree is hatted in ω. In the above example we have $\omega_1 < \omega_2 < \ldots < \omega_8$.

6.4.5 Lemma. *The set Ω of complete words partitions X and E into subsets X^ω and E^ω where*

$$X^\omega = \{T_R \in X \; ; \; i \in \omega \Rightarrow i \in R, \; \hat{i} \in \omega \Rightarrow i \notin R\}$$

$$E^\omega = \{E_P \in E \; ; \; i \in \omega \Rightarrow i \in P, \; \hat{i} \in \omega \Rightarrow i \notin P\}.$$

Proof. Let $\omega, \omega' \in \Omega$ with $X^\omega \cap X^{\omega'} \neq \emptyset$. Let $T_R \in X^\omega \cap X^{\omega'}$. Then R satisfies the conditions of both ω and ω', that is, whenever i is in ω or ω' then $i \in R$, and whenever $\hat{i}$ is in ω or ω' then $i \notin R$. For $j \in \{1, \ldots, n\}$ we say the j^{th} entry of a word is the entry equal to j or $\hat{j}$, if such an entry exists. The n^{th} entry of a word always exists. If ω and ω' both have a j^{th} entry, for any $j \in \{1, \ldots, n\}$, then both entries must equal j if $j \in R$, and both entries must equal $\hat{j}$ if $j \notin R$, and in particular, both entries must be equal. Therefore, one of ω and ω' must be an initial segment of the other, and $\omega_1 = \omega_2$ by the definition of a complete word.

Let $T_R \in X$. Define the word ω_0 to be the sequence $(a_n, \ldots, a_1)$, where $a_j = j$ if $j \in R$ and $a_j = \hat{j}$ if $j \notin R$. Since $|R| = \rho > \rho - 1$ some initial segment of ω_0 is of type 1. Let ω be the smallest initial segment of ω_0 which is of type 1 or type 2. Then ω is a complete word with $T_R \in X^\omega$.

A similar argument works for E.

□

6.4.6 Example. Let $n = 7$, $m = 6$ and therefore $\rho = 3$. The set of complete words w_1, ..., w_8 is listed in Example 6.4.4, and we will describe the partitions of X and E these define.

A tableau in X is a (1^6)-tableau with all entries repeated. To make the example clearer we abbreviate the tableaux in X as follows. For the (1^6)-tableau

$$\begin{array}{|c|}\hline i_1 \\ \hline \bar{i}_1 \\ \hline i_2 \\ \hline \bar{i}_2 \\ \hline i_3 \\ \hline \bar{i}_3 \\ \hline \end{array}$$

we write

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline \end{array}\,.$$

An expansion in E is of the form $E_{\{j_1,j_2\}} = E_1\left(\begin{array}{|c|}\hline j_1 \\ \hline \bar{j}_1 \\ \hline j_2 \\ \hline \bar{j}_2 \\ \hline \end{array}\right)$. Instead of this we write $E_{\begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \end{array}}$.

Then

$$X^{\omega_1} = \left\{ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 4\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 4\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array} \right\}$$

$$X^{\omega_2} = \left\{ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 6\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 6\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 6\\\hline\end{array} \right\} \quad X^{\omega_3} = \left\{ \begin{array}{|c|}\hline 1\\\hline 4\\\hline 6\\\hline\end{array} \right\} \quad X^{\omega_4} = \left\{ \begin{array}{|c|}\hline 1\\\hline 5\\\hline 6\\\hline\end{array} \right\}$$

$$X^{\omega_5} = \left\{ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 7\\\hline\end{array} \right\} \quad X^{\omega_6} = \left\{ \begin{array}{|c|}\hline 1\\\hline 4\\\hline 7\\\hline\end{array} \right\} \quad X^{\omega_7} = \left\{ \begin{array}{|c|}\hline 1\\\hline 5\\\hline 7\\\hline\end{array} \right\}$$

$$X^{\omega_8} = \left\{ \begin{array}{|c|}\hline 1\\\hline 6\\\hline 7\\\hline\end{array} \right\}$$

$$E^{\omega_1} = \left\{ E_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 2\\\hline 5\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}} \right\}$$

$$E^{\omega_2} = \left\{ E_{\begin{array}{|c|}\hline 1\\\hline 6\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 2\\\hline 6\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 3\\\hline 6\\\hline\end{array}} \right\} \quad E^{\omega_3} = \left\{ E_{\begin{array}{|c|}\hline 4\\\hline 6\\\hline\end{array}} \right\} \quad E^{\omega_4} = \left\{ E_{\begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array}} \right\}$$

$$E^{\omega_5} = \left\{ E_{\begin{array}{|c|}\hline 1\\\hline 7\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 2\\\hline 7\\\hline\end{array}}, E_{\begin{array}{|c|}\hline 3\\\hline 7\\\hline\end{array}} \right\} \quad E^{\omega_6} = \left\{ E_{\begin{array}{|c|}\hline 4\\\hline 7\\\hline\end{array}} \right\} \quad E^{\omega_7} = \left\{ E_{\begin{array}{|c|}\hline 5\\\hline 7\\\hline\end{array}} \right\}$$

$$E^{\omega_8} = \left\{ E_{\begin{array}{|c|}\hline 6\\\hline 7\\\hline\end{array}} \right\}$$

We construct a total ordering on the sets X and E as follows. Given two distinct repeat sets $R_1, R_2 \subset I$ (or two distinct equation sets $P_1, P_2 \subset I$) then $R_1 < R_2$ ($P_1 < P_2$ respectively) if the largest number contained in exactly one of them is contained in R_2 (respectively P_2). For example, $\{1,3,5\} < \{1,4,5\}$. This gives an ordering on X and E by saying that $T_{R_1} < T_{R_2}$ if and only if $R_1 < R_2$ and $E_{P_1} < E_{P_2}$ if and only if $P_1 < P_2$.

In the above example, the elements of the sets $X^{\omega_1}, \ldots, X^{\omega_8}$ were written in increasing order, not only within the sets, but from set to set. The same is true for the sets $E^{\omega_1}, \ldots, E^{\omega_8}$.

6.4.7 Lemma. *Let ω_1 and ω_2 be complete words with $\omega_1 < \omega_2$. If $T_{R_1} \in X^{\omega_1}$ and $T_{R_2} \in X^{\omega_2}$ then $T_{R_1} < T_{R_2}$, and if $E_{P_1} \in E^{\omega_1}$ and $E_{P_2} \in E^{\omega_2}$ then $E_{P_1} < E_{P_2}$.*

Proof. If the largest number on which ω_1 and ω_2 disagree is i then the largest number on which R_1 and R_2 disagree is also i. Since i is hatted in ω_1 and unhatted in ω_2, $i \notin R_1$ and $i \in R_2$. So $T_{R_1} < T_{R_2}$. The same observation shows $E_{P_1} < E_{P_2}$.

□

6.4.8 Definitions. Let $[X]$ be the column matrix obtained by putting a bideterminant in each row, one for each λ-tableau in X, in increasing order down the column.

Then the rows of $[X]$ are indexed by the set of non-symplectic sets $R \subset I$ satisfying $|R| = \rho = \frac{m}{2}$.

Let $P \subset I$ with $|P| = \rho - 1$. Then the equation $E_P = 0$ is of the form

$$\sum_{R \subset I, |R|=\rho, P \subset R} D_{T_R} = 0$$

which is equivalent to

$$\sum_{\substack{R \subset I, |R|=\rho, P \subset R \\ R \text{ non-symplectic}}} D_{T_R} = \sum_{\substack{R \subset I, |R|=\rho, P \subset R \\ R \text{ symplectic}}} -D_{T_R}.$$

Let $[E]$ be a column matrix with one row for each expansion $E_P \in E$ in increasing order. Let the row corresponding to E_P contain the right hand side of the equation of the form above obtained from $E_P = 0$. Then the rows of $[E]$ are indexed by the subsets $P \subset I$ with $|P| = \rho - 1$, and the row corresponding to P contains the expression obtained by removing from E_P all bideterminants of non-symplectic tableaux, and multiplying the remaining terms by -1.

Let M be an $|E| \times |X|$ matrix with rows indexed by the set of $P \subset I$ satisfying $|P| = \rho - 1$, in increasing order, and columns indexed by the set of non-symplectic sets $R \subset I$ satisfying $|R| = \rho$, in increasing order. Let the $(P, R)^{\text{th}}$ entry of M be the coefficient of D_{T_R} in E_P. Then the entries in M are either 0 or 1. In fact,

$$M_{P,R} = \begin{cases} 1 & P \subset R \\ 0 & P \not\subset R \,. \end{cases}$$

Then all the equations $E_P = 0$ where $E_P \in E$ can be expressed by the following matrix equation.

$$M[X] = [E].$$

We aim to show that M is a square matrix which is invertible when the characteristic of $\mathbf{k}$ is zero or sufficiently large. This will show that the equations $E_P = 0$ with $|P| = \rho - 1$ suffice to express all bideterminants D_{T_R} with $|R| = \rho$ as a linear combination of bideterminants of symplectic tableaux.

We have a decomposition of X into subsets X^ω, and a decomposition of E into subsets E^ω. There is a corresponding decomposition of the matrix M into blocks M^ω where M^ω is the submatrix of M which is the intersection of the rows indexed by P where $E_P \in E^\omega$ and the columns indexed by R where $T_R \in X^\omega$.

6.4.9 Example. Let $n = 4$ and $m = 4$. So $\rho = 2$ and $n - 2\rho + 1 = 1$. The complete words are $\omega_1 = (\hat{4})$ and $\omega_2 = (4)$ with $\omega_1 < \omega_2$.

$$
\begin{aligned}
X^{\omega_1} &= \left\{ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \right\}\\
X^{\omega_2} &= \left\{ \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} \right\}\\
E^{\omega_1} &= \left\{ E_{\boxed{1}}, E_{\boxed{2}}, E_{\boxed{3}} \right\}\\
E^{\omega_2} &= \left\{ E_{\boxed{4}} \right\}.
\end{aligned}
$$

The relations are as follows

$$
\begin{aligned}
E_{\boxed{1}} &= D_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}},\\
E_{\boxed{2}} &= D_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}},\\
E_{\boxed{3}} &= D_{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}},\\
E_{\boxed{4}} &= D_{\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}},
\end{aligned}
$$

and so M is of the form

$$
\begin{array}{c}
\\ \{1\} \\ \{2\} \\ \{3\} \\ \{4\}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \{1,2\} & \{1,3\} & \{2,3\} & \{1,4\} \end{array}\\
\left(\begin{array}{cccc}
1 & 1 & 0 & 1\\
1 & 0 & 1 & 0\\
0 & 1 & 1 & 0\\
0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

with

$$
M^{\omega_1} = \begin{pmatrix} 1\,1\,0\\ 1\,0\,1\\ 0\,1\,1 \end{pmatrix} \text{ and } M^{\omega_2} = (\,1\,).
$$

6.4.10 Definition. Define the *length* of a word to be the number of entries. So when $w = (a_n, a_{n-1}, \ldots, a_l)$ the length is given by $l(w) = n - l + 1$.

6.4.11 Lemma. *The matrix M is triangular in block form, and has one diagonal block M^ω corresponding to each complete word ω. M is of the form*

$$\begin{array}{cc} & \overset{X^\omega}{\longleftrightarrow} \\ E^\omega \updownarrow & \begin{pmatrix} \ddots & & & & \\ & \boxed{*} & & * & \\ & & \boxed{M^\omega} & & \\ & 0 & & \boxed{*} & \\ & & & & \ddots \end{pmatrix} \end{array}$$

Proof. Let ω be a complete word. We first show that $|E^\omega| = |X^\omega|$, and hence that M^ω is square.

Let ω be of type 1. There are $\rho - 1$ unhatted numbers in ω, let them be $\omega_1, \ldots, \omega_{\rho-1}$ with $\omega_1 < \ldots < \omega_{\rho-1}$. The set E^ω contains only one element, and that is E_P where $P = \{\omega_1, \ldots, \omega_{\rho-1}\}$. So $|E^\omega| = 1$. If $T_R \in X^\omega$ then R must contain $\omega_1, \ldots, \omega_{\rho-1}$. Since no initial segment of ω is of type 2 we have $\omega_i \geq 2(i+1)$ for all $i \in \{1, \ldots, \rho - 1\}$. If the remaining element of R is chosen to be not less than 2 then R will be symplectic. Hence the only non-symplectic set of the required form is $R = \{1, \omega_1, \ldots, \omega_{\rho-1}\}$ and $|X^\omega| = 1$. Hence $|E^\omega| = |X^\omega| = 1$.

Now suppose that ω is of type 2, and let μ denote the number of unhatted elements in ω. Then the length of ω, denoted by $\ell(\omega)$, is $2\mu + n - 2\rho + 1$. For any $E_P \in E^\omega$ all but $\rho - \mu - 1$ of the elements of P are determined by ω. The last elements can be any choice from the set $\{1, \ldots, 2\rho - 2\mu - 1\}$, and so

$$|E^\omega| = \binom{2\rho - 2\mu - 1}{\rho - \mu - 1} = \binom{2\rho - 2\mu - 1}{\rho - \mu}.$$

If $T_R \in X^\omega$ then μ of the elements of R are determined. Call these fixed elements $r_{\rho-\mu+1}, \ldots, r_\rho$, with $r_{\rho-\mu+1} < \ldots < r_\rho$. Since $\ell(\omega) = 2\mu + n - 2\rho + 1$ the remaining $\rho - \mu$ elements of R can be chosen from the set $\{1, \ldots, 2\rho - 2\mu - 1\}$. Since all of these are smaller than the fixed elements $r_{\rho-\mu}$ will be chosen from this set and so all choices give $r_{\rho-\mu} < 2(\rho - \mu)$. So all possible choices for the remaining $\rho - \mu$ elements of R will result in R being non-symplectic. Hence

$$|X^\omega| = \binom{2\rho - 2\mu - 1}{\rho - \mu}.$$

and M^ω is square for any complete word ω.

Consider an entry $M_{P,R}$ below the diagonal blocks. If $E_P \in E^{\omega_1}$ and $T_R \in X^{\omega_2}$ then $\omega_1 > \omega_2$, and there is an element i which is hatted in ω_2 and unhatted in ω_1. Hence $i \in P$ but $i \notin R$, and P cannot be a subset of R. Therefore the bideterminant of T_R does not occur in the expansion E_P, and $M_{P,R} = 0$.

□

6.4.12 Corollary. *$|E| = |X|$ and M is a square matrix.*

□

To show a solution to $M[X] = E$ exists for $[X]$ it is sufficient to show M is invertible.

6.4.13 Lemma. *Let $\omega \in \Omega$ If* $\operatorname{char}(\mathbf{k}) = 0$ *or* $\operatorname{char}(\mathbf{k}) > \rho$ *then M^ω is invertible.*

Proof. When ω is of type 1 $M^\omega = (1)$ which is invertible.

When ω is of type 2 we know that

$$(M^\omega)_{P,R} = \begin{cases} 1 & P \subset R \\ 0 & P \not\subset R \end{cases}$$

where $E_P \in E^\omega$ and $T_R \in X^\omega$. Let μ be the number of unhatted elements in ω, so $\ell(\omega) = 2\mu + n - 2\rho + 1$, and let $\nu = \rho - |R \cap P|$, so when $P \subset R$ we have $\nu = 1$.

Let N^ω be a matrix with rows indexed by non-symplectic subsets $R \subset I$ with $|R| = \rho$ and columns indexed by the subsets $P \subset I$ satisfying $|P| = \rho - 1$, both in increasing order. Let the $(R,P)^{\text{th}}$ coefficient of N^ω be given by

$$(N^\omega)_{R,P} = \frac{(-1)^{\nu-1}}{\binom{\rho-\mu}{\nu}\nu}$$

Since $1 \le \nu \le \rho - \mu$ we have $\nu \le \rho$ and $\binom{\rho-\mu}{\nu} \le \rho$, and as long as $\operatorname{char}(\mathbf{k}) > \rho$ or $\operatorname{char}(\mathbf{k}) = 0$ we are not attempting to divide by zero in the above expression.

We wish to show that N^ω is the inverse of M^ω. Consider the product $M^\omega N^\omega$. Let E_{P_1} and E_{P_2} be elements of E^ω. Then

$$(M^\omega N^\omega)_{P_1,P_2} = \sum_{T_R \in X^\omega} (M^\omega)_{P_1,R}(N^\omega)_{R,P_2} = \sum_{T_R \in X^\omega \text{ s.t. } P_1 \subset R} (N^\omega)_{R,P_2}.$$

If $P_1 \neq P_2$ then $\mu \le |P_1 \cap P_2| < \rho - 1$. Let $|P_1 \cap P_2| = \rho - \nu$ for some $\nu \in \{2, \ldots, \rho - \mu\}$. Since $P_1 \subset R$ all but one of the elements of R are determined, and since $T_R \in X^\omega$ and ω has μ unhatted entries, there are $\rho - \mu$ possible choices $\{i_1, i_2, \ldots, i_{\rho-\mu}\}$ for the last element. Now $\nu - 1$ of $\{i_1, i_2, \ldots, i_{\rho-\mu}\}$ are in P_2 since $|P_1 \cap P_2| = \rho - \nu$, and so there are $\nu - 1$ tableaux $T_R \in X^\omega$ such that $P_1 \subset R$ and $|R \cap P_2| = \rho - \nu + 1$. This leaves $\rho - \mu - \nu + 1$ tableaux $T_R \in X^\omega$ such that $P_1 \subset R$ and $|R \cap P_2| = \rho - \nu$. So

$$\begin{aligned}(M^\omega N^\omega)_{P_1P_2} &= \frac{(\nu-1)(-1)^{\nu-2}}{\binom{\rho-\mu}{\nu-1}(\nu-1)} + \frac{(\rho-\mu-\nu+1)(-1)^{\nu-1}}{\binom{\rho-\mu}{\nu}\nu} \\ &= \frac{(-1)^{\nu-2}}{\binom{\rho-\mu}{\nu-1}} + \frac{(-1)^{\nu-1}}{\binom{\rho-\mu}{\nu-1}} = 0.\end{aligned}$$

Alternatively, if $P_1 = P_2$ then whenever $P_1 \subset R$ we have $P_2 \subset R$. The number of $T_R \in X^\omega$ such that $P_1 \subset R$ is $\rho - \mu$. So

$$(M^\omega N^\omega)_{P_1P_2} = \frac{(\rho-\mu)(-1)^0}{\binom{\rho-\mu}{1}} = 1.$$

Hence N^ω is the inverse of M^ω.

□

So M is invertible, and M^{-1} is of the form

$$\begin{array}{cc} & \overset{E^\omega}{\longleftrightarrow} \\ X^\omega \updownarrow & \begin{pmatrix} \ddots & & & & \\ & \boxed{*} & & * & \\ & & \boxed{N^\omega} & & \\ & 0 & & \boxed{*} & \\ & & & & \ddots \end{pmatrix} \end{array}.$$

6.4.14 Example When $n = 4$ and $\rho = 2$, as in Example 6.4.9, as long as $\text{char}(\mathbf{k}) = 0$ or $\text{char}(\mathbf{k}) > 2$ we have

$$M^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Case when $S \neq \emptyset$.

Let $S \subset I \cup \overline{I}$ with $S = \{s_1, \ldots, s_\varsigma\}$ such that $m - \varsigma$ is even. Let X^S denote the set of all non-symplectic semistandard λ-tableaux with single set S. Then $m = 2\rho + \varsigma$ where ρ is the length of the repeat sets of the tableaux in X^S. Let $mod(S) = \{|s_1|, \ldots, |s_\varsigma|\}$, where $|i| = i$ and $|\overline{i}| = i$ for all $i \in I$. Then $mod(S) \subset I$. If $\varsigma = m$ then $X^S = \emptyset$ because a tableau with no repeats is symplectic. So assume $\varsigma < m$ and $\rho > 0$. Let E^S be the set of all expansions $E_{P,S}$ with $P \subset I$ such that $|P| = \rho - 1$ and $P \cap mod(S) = \emptyset$. If a bideterminant of a tableau in X^S appears in a simple expansion, then that expansion is in the set E^S.

6.4.15 Definition. $R \subset I$ is S-symplectic if $R \cap mod(S) = \emptyset$ and $T_{R,S}$ is symplectic.

Hence X^S is indexed by all sets $R \subset I$ such that $|R| = \rho$ and R is S-symplectic.

6.4.16 Definition. S and n determine a map ${}_n\Upsilon_S : \{1, \ldots, n\} \setminus mod(S) \to \{1, \ldots, n - \varsigma\}$ given by

$$ {}_n\Upsilon_S(i) = i - t_i $$

where $t_i = |\{s \in mod(S); s < i\}|$.

The function of ${}_n\Upsilon_S$ is to remove the elements of $mod(S)$ from $\{1, \ldots, n\}$ and rename the remaining elements $\{1, \ldots, n - \varsigma\}$ in the same order. The map ${}_n\Upsilon_S$ can be extended to subsets of I by defining

$$ {}_n\Upsilon_S(\{i_1, i_2, \ldots, i_p\}) = \{{}_n\Upsilon_S(i_1),\ {}_n\Upsilon_S(i_2), \ldots,\ {}_n\Upsilon_S(i_p)\} $$

for any $i_1, i_2, \ldots, i_p \in I$.

6.4.17 Lemma. *Let $R \subset I$ be such that $R \cap mod(S) = \emptyset$. Then R is S-symplectic if and only if ${}_n\Upsilon_S(R)$ is symplectic.*

Proof. R is S-symplectic if and only if the entry in the i^{th} row of $T_{R,S}$ is not less than i, for $i = 1, \ldots, m$. If the i^{th} row contains an element of S, that is, an unrepeated element, then it will be less than i if and only if the element above is less than $i-1$, because it must have a modulus at least one greater then the element above. So we need only consider the rows containing repeated elements.

Let the entries in rows $i-1$ and i of $T_{R,S}$ be r_j and $\overline{r}_j$ respectively, where $j \in \{1, \ldots, \rho\}$. Since $T_{R,S}$ is semistandard the entries $r_1, \ldots, r_{j-1}, \overline{r}_1, \ldots, \overline{r}_{j-1}$ all occur in the rows above row i, and also all the $s \in S$ such that $s < r_j$ occur in the rows above row i. Hence $i = 2(j-1) + t_{r_j} + 2 = 2j + t_{r_j}$. So R is S-symplectic if and only if $r_j \geq 2j + t_{r_j}$, that is, $r_j - t_{r_j} \geq 2j$ for all $j \in \{1, \ldots, \rho\}$. This is exactly the condition under which ${}_n\Upsilon_S(R)$ is symplectic.

□

The map ${}_n\Upsilon_S$ extended to sets in this way gives a bijection between X^S and the set X' of non-symplectic semistandard $(1^{2\rho})$-tableaux $T_{R'}$ with $R' \subset \{1, \ldots, n - \varsigma\}$ and $|R'| = \rho$. It also gives a bijection between E^S and the set E' of simple expansions involving one or more bideterminants of tableaux in X'. Using the expansions in E' we construct a matrix M' as before with $M'[X'] = [E']$. M' is the analogue to the matrix M, of the last section, in the case of $Sp_{2(n-\varsigma)}(\mathrm{k})$ for the partition $(1^{2\rho})$, since the tableaux in X' contain only repeated elements. Hence we already know that M' is invertible as long as $\mathrm{char}(\mathrm{k}) > \rho$.

Let M^S be the matrix whose elements are the coefficients of the bideterminants of the tableaux in X^S in the expansions in E^S. Let $[E^S]$ be the column vector containing in row P the negative sum of the symplectic bideterminants in $E_{P,S}$, and let $[X^S]$ be the column vector containing bideterminants of the elements $T_{R,S} \in X^S$, both in increasing order according to the orderings on the sets P and R respectively. Then $M^S[X^S] = [E^S]$ gives the relations $E_{P,S} = 0$, in matrix form, where $E_{P,S} \in E^S$.

6.4.18 Lemma. $M^S = M'$.

Proof. Since ${}_n\Upsilon_S$ is a bijective map from subsets of I to subsets of $\{1,\ldots,n-\varsigma\}$, which preserves the property of being symplectic, it is sufficient to show that if

$$E_{P,S} = D_{T_{R_1},S} + D_{T_{R_2},S} + \ldots + D_{T_{R_n},S}$$

and we write $R'_i = {}_n\Upsilon_S(R_i)$ for $i = 1,\ldots,n$ then the expression

$$D_{T_{R'_1}} + D_{T_{R'_2}} + \ldots + D_{T_{R'_n}} = 0$$

is equal to $E_{{}_n\Upsilon_S(P)}$.

In the expression for $E_{P,S}$, $R_i = P \cup \{i\}$ for $i = 1,..,n$ rearranged into increasing order. Since the bideterminant of a tableau containing the same entry twice in a column is zero, if $s \in S$ then $D_{T_{R_i},S} = 0$. So

$$E_{P,S} = D_{T_{R_{i_1}},S} + \ldots + D_{T_{R_{i_{n-\varsigma}}},S}$$

where $\{i_1,\ldots,i_{n-\varsigma}\} = \{1,\ldots,n\} \setminus mod(S)$. Then ${}_n\Upsilon_s(\{i_1,\ldots,\complement_{n-\varsigma}\}) = (\{1,\ldots,n-\varsigma\})$, and ${}_n\Upsilon_s(R_{i_j}) = R'_j$ for $j \in \{1,\ldots,n-\varsigma\}$ where $R'_j = P' \cup \{j\}$ and $P' = {}_n\Upsilon_s(P)$. Therefore

$$E_{P'} = D_{T_{R'_1}} + \ldots + D_{T_{R'_{n-\varsigma}}}$$

as required.

□

6.4.19 Corollary. *If the* char(k) $= 0$ *or* char(k) $= p > \rho$ *then* M^S *is invertible.*

□

6.4.20 Lemma. *Let T be a (1^m)-tableau for some $m \leq n$. Let ρ be the length of the repeat set of T. If* char(k) $= 0$ *or* char(k) $> \rho$ *then D_T can be expressed as a linear combination of bideterminants of symplectic λ-tableaux.*

Proof. If T is symplectic then the above is clear. If T is not symplectic then T is an element of the set X^S of non-symplectic (1^m)-tableaux with single set S, and $T = T_{R,S}$ where R is not S-symplectic. Since char(**k**) $= 0$ or char(**k**) $> \rho$ then M^S is invertible and

$$[X^S]_R = ((M^S)^{-1}[E^S])_R$$

gives the required expression for $D_T = D_{T_{R,S}}$.

□

The expression for $D_{T_{R,S}}$ obtained in this way is called its matrix solution. Notice that when char(**k**) $= 2$ then the inverse M^{-1} of Example 6.4.14 does not exist. So this method only works if char(**k**) $= 0$ or char(**k**) is sufficiently large.

6.5 The Case when λ has more than one column.

In order to use the previous results for column tableaux here, we need the following.

6.5.1 Lemma. *Let $m \in \mathbb{Z}$ with $m \leq n$, and let $T_{R,S}$ be a non-symplectic semistandard (1^m)-tableau. If $D_{T_{R',S}}$ occurs in the matrix solution of $D_{T_{R,S}}$ then $R < R'$.*

Proof. Since ${}_n\Upsilon_S$ preserves the ordering on repeat sets we can assume $S = \emptyset$. D_{T_R} is in the R-row of $[X]$, and $[X] = M^{-1}[E]$ gives

$$D_{T_R} = \sum_{E_P \in E} (M^{-1})_{R,P}[E]_P.$$

Let $T_R \in X^\omega$. Whenever $E_P \in E^{\omega'}$ with $\omega' < \omega$ we have $(M^{-1})_{R,P} = 0$. Consider $E_P \in E^{\omega'}$ with $\omega < \omega'$. $[E]_P$ contains the negative sum of the bideterminants of the symplectic (1^m)-tableaux which occur in the expansion E_P. So if $D_{T_{R'}}$ is a term in $[E]_P$ then $P \subset R'$. Let the largest element on which ω and ω' differ be i. Then i is hatted in ω, so $i \notin R$, and i is unhatted in ω' so $i \in R'$. Since $T_R \in X^\omega$ any elements greater than i which are contained in R must be unhatted in ω, and thus in ω', and so must be contained in R'. Hence the largest element on which R and R' disagree is i and $R < R'$.

Now let $E_P \in E^\omega$. If ω is of type 1 then R contains 1 and the unhatted elements of ω. If $D_{T_{R'}}$ occurs in $[E]_P$ then R' contains all the unhatted elements of ω and one member of $\{2, \ldots, n\}$. Therefore $R < R'$.

Finally, suppose ω is of type 2. Then ω determines whether or not each element of $\{n - \ell(\omega) + 1, \ldots, n-1, n\}$ is contained in R and P. Since ω has $n - m + 1$ more hatted than unhatted elements, these requirements force R to be non-symplectic. If $D_{T_{R'}}$ occurs in $[E]_P$ then $P \subset R'$, and all the unhatted elements of ω are in R'. However, since R' is symplectic, at least one of the hatted elements of ω must also be in R'. This will be the largest element on which R and R' disagree, and hence $R < R'$.

So whenever $(M^{-1})_{R,P} \neq 0$, E_P contains only bideterminants $D_{T_{R'}}$ which satisfy $R < R'$.

□

6.5.2 Definitions. We define an equivalence relation $\sim$ on λ-tableaux with entries from $I \cup \bar{I}$. If T and T' are two such tableaux then $T \sim T'$ if the entries $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ all occur with the same multiplicity in T as in T'. Let (T) denote the equivalence class of T.

Let $I \cup \bar{I}$ be ordered via $1 < \bar{1} < 2 < \bar{2} < \ldots < n < \bar{n}$. Then there is a total ordering on these equivalence classes. If $(T) \neq (T')$ then $(T) < (T')$ if the largest entry which does not occur with the same multiplicity in T as in T' occurs with greater multiplicity in T'.

Let λ be the partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Let T be a semistandard non-symplectic λ-tableau. By the definition of a bideterminant

$$D_T = D_{T_1} D_{T_2} \ldots D_{T_{\lambda_1}}$$

where $T_1, \ldots, T_{\lambda_1}$ are the columns of T. By substituting the matrix solutions for the columns $T_1, T_2, \ldots, T_{\lambda_1}$ of T into this expression we obtain the matrix solution for D_T

$$D_T = k_1 D_{T'_1} + k_2 D_{T'_2} + \ldots + k_s D_{T'_s}$$

where $k_1, k_2, \ldots, k_s \in \mathbf{k}$ and $T'_1, T'_2, \ldots, T'_s$ are λ-tableaux which are not necessarily semistandard, but which have columns which are symplectic considered on their own.

6.5.3 Lemma. *Let T be a non-symplectic semistandard λ-tableau, for which a matrix solution*

$$D_T = k_1 D_{T'_1} + k_2 D_{T'_2} + \ldots + k_s D_{T'_s}$$

exists, where $k_1, \ldots, k_s \in \mathbf{k}$. Then $(T) < (T'_i)$ for all $i \in \{1, \ldots, s\}$.

Proof. At least one column of T is non-symplectic when considered on its own. Let the j^{th} column be so. Let $R'_{i,j}$ be the repeat set of the j^{th} column of T'_i, and let R_j be the repeat set of the j^{th} column of T. By the last lemma $R_j < R'_{i,j}$, and if $l_j \in I \cup \bar{I}$ is the largest number on which they differ, $l_j \in R'_{i,j}$ but $l_j \notin R_j$. Let $l = \max\{l_1, \ldots, l_{\lambda_1}\}$. Then the multiplicity of any element of $I \cup \bar{I}$ larger than l is the same in T as in T'_i, but l occurs at least once more in T'_i than in T. So $(T) < (T'_i)$.

□

We have not shown that this expression is unique, so we will refer to an expression obtained in this way as the matrix solution for $D_{T_{R,S}}$.

Let μ be the conjugate partition to λ.

6.5.4 Proposition. *Let T be a semistandard non-symplectic λ-tableau. If* $\operatorname{char}(\mathbf{k}) = 0$ *or* $\operatorname{char}(\mathbf{k}) > \frac{\mu_1}{2}$ *then D_T can be expressed as a linear combination of bideterminants of symplectic λ-tableaux, making use only of the simple expansion operators.*

Proof. The length of any column of T is not more than μ_1, and so its repeat set has length not more than $\frac{\mu_1}{2}$. Hence the matrix solution for D_T exists. Let

$$D_T = k_1 D_{T'_1} + k_2 D_{T'_2} + \ldots + k_s D_{T'_s}$$

be the matrix solution for D_T.

Each T'_i in the matrix solution for T has symplectic columns, when considered on their own. However, the T'_i are not necessarily semistandard since they may not be non-decreasing along their rows. Carter and Lustzig in [1] have given a method for expressing the bideterminant of any λ-tableau as a linear combination of bideterminants of semistandard λ-tableaux. These semistandard tableaux are obtained by permuting the entries of the original tableau, and so are in the same equivalence class as the original.

By using their method we obtain an expression for each $D_{T'_i}$ in terms of bideterminants of semistandard λ-tableaux, all in the same equivalence class as T'_i. These are no longer necessarily symplectic, and for each T'_i which is not symplectic, the whole process can be repeated. However, each time the process is repeated the λ-tableaux produced belong to a higher equivalence class. Since there are only finitely many λ-tableaux, and hence only finitely many equivalence classes of them, the process must terminate, and we obtain the desired expression.

□

6.6 The Spanning Property in General.

We wish to prove that the set of bideterminants D_T as T ranges over all symplectic λ-tableaux. spans $D^{sp}_{\lambda,\mathbf{k}}$ when the characteristic of $\mathbf{k}$ is arbitrary, and we use the expansion operators to do this. The last section was not necessary for the general result, but we have shown that when the characteristic of $\mathbf{k}$ is zero or sufficiently large, only the simple expansions are necessary. In order to go further we need more information than is given by the simple relations between bideterminants.

6.6.1 Definition. Let $m, t \in \mathbb{N}$ with $m + 2t \leq n$, and let $\alpha_1, \ldots, \alpha_m \in I \cup \bar{I}$. Let T be the (1^m)-tableau of the form $T = \begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \vdots \\ \hline \alpha_m \\ \hline \end{array}$. We define a map

$$w_t^+ : D^{sp}_{(1^m),k} \rightarrow D^{sp}_{(1^{m+2t}),k}$$

to be the linear extension of

$$w_t^+(D_T) = \sum_{\substack{i_1, i_2, \ldots, i_t \in I \\ i_1 < i_2 < \ldots < i_t}} D_{T(i_1, \bar{i}_1, i_2, \bar{i}_2, \ldots, i_t, \bar{i}_t)}$$

where $T(i_1, \overline{i_1}, \ldots, i_t, \overline{i_t})$ denotes the (1^{m+2t})-tableau $\begin{array}{|c|}\hline \alpha_1\\\hline \vdots\\\hline \alpha_m\\\hline i_1\\\hline \overline{i_1}\\\hline \vdots\\\hline i_t\\\hline \overline{i_t}\\\hline\end{array}$.

6.6.2 Example. Let $n = 6$, $m = 2$, $t = 2$ and $T = \begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline\end{array}$. Then

$$w_2^+(D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline\end{array}}) = D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 2\\\hline \overline{2}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 3\\\hline \overline{3}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 2\\\hline \overline{2}\\\hline 3\\\hline \overline{3}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 2\\\hline \overline{2}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 2\\\hline \overline{2}\\\hline 5\\\hline \overline{5}\\\hline\end{array}}$$

$$+ D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 2\\\hline \overline{2}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 3\\\hline \overline{3}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 3\\\hline \overline{3}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 3\\\hline \overline{3}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 4\\\hline \overline{4}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 4\\\hline \overline{4}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 5\\\hline \overline{5}\\\hline 6\\\hline \overline{6}\\\hline\end{array}}$$

Since the determinant of a matrix with two equal rows is zero, the above is equal to

$$D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 3\\\hline \overline{3}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 1\\\hline \overline{1}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 3\\\hline \overline{3}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 3\\\hline \overline{3}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 5\\\hline \overline{5}\\\hline 6\\\hline \overline{6}\\\hline\end{array}}.$$

Rearranging the columns into increasing order involves only even permutations of the rows, and so

$$w_s^+(D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline\end{array}}) = D_{\begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline 2\\\hline 3\\\hline \overline{3}\\\hline 4\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline 2\\\hline \overline{4}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline 2\\\hline \overline{4}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline 3\\\hline \overline{3}\\\hline \overline{4}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline 3\\\hline \overline{3}\\\hline \overline{4}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline 5\\\hline \overline{5}\\\hline 6\\\hline \overline{6}\\\hline\end{array}}.$$

In order to make notation easier when handling determinants we define

$$D^{a_1,\ldots,a_s}_{b_1,\ldots,b_s} = \begin{vmatrix} d_{a_1,b_1} & \ldots & d_{a_1,b_s} \\ \vdots & & \vdots \\ d_{a_s,b_1} & \cdots & d_{a_s,b_s} \end{vmatrix} \in A^{sp}(\overline{n})$$

for all $a_1,\ldots,a_s,b_1,\ldots,b_s \in I \cup \overline{I}$ and $s \in \mathbb{N}$.

6.6.3 Example. The quadratic form $Q_{i,j}$ can be written as

$$Q_{i,j} = \sum_{k=1}^{n} D^{i,j}_{k,\overline{k}}.$$

The Laplace Determinant Expansion.
Let $E = (e_{i,j})$ be any $s \times s$ square matrix for $s \in \mathbb{N}$. Then for any $t \in \mathbb{N}$ the Laplace expansion of determinants applied to columns gives the following.

$$\det E = \sum_{\substack{i_1,\ldots,i_t \in \{1,\ldots,s\} \\ i_1 < i_2 < \ldots < i_t}} \operatorname{sign}(\sigma) \begin{vmatrix} e_{i_1,1} & \ldots & e_{i_1,t} \\ \vdots & & \vdots \\ e_{i_t,1} & \cdots & e_{i_t,t} \end{vmatrix} \begin{vmatrix} e_{j_1,t+1} & \ldots & e_{j_1,s} \\ \vdots & & \vdots \\ e_{j_{s-t},t+1} & \cdots & e_{j_{s-t},s} \end{vmatrix}$$

summed over all choices of t rows $i_1,\ldots,i_t$ of E with $i_1 < \ldots < i_t$, and where $j_1,\ldots,j_{s-t}$ are the remaining rows with $j_1 < \ldots < j_{s-t}$ and

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & t & t+1 & t+2 & \ldots & s \\ i_1 & i_2 & \ldots & i_t & j_1 & j_2 & \ldots & j_{s-t} \end{pmatrix}.$$

6.6.4 Lemma. *Let $t \in \mathbb{N}$ with $2t < n$, and let $a_1,\ldots,a_t,b_1,\ldots,b_t \in I$ be distinct. Then*

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_t=1}^{n} D^{a_1,b_1}_{i_1,\overline{i_1}} D^{a_2,b_2}_{i_2,\overline{i_2}} \ldots D^{a_t,b_t}_{i_t,\overline{i_t}} = 0.$$

Proof. Since

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_t=1}^{n} D^{a_1,b_1}_{i_1,\overline{i_1}} D^{a_2,b_2}_{i_2,\overline{i_2}} \ldots D^{a_t,b_t}_{i_t,\overline{i_t}} = \sum_{i_1=1}^{n} D^{a_1,b_1}_{i_1,\overline{i_1}} \left(\sum_{i_2=1}^{n} \cdots \sum_{i_t=1}^{n} D^{a_2,b_2}_{i_2,\overline{i_2}} \ldots D^{a_t,b_t}_{i_t,\overline{i_t}} \right),$$

the lemma follows by observing that the quadratic relation $Q_{a_1,b_1} = 0$ gives

$$\sum_{i_1=1}^{n} D^{a_1,b_1}_{i_1,\overline{i_1}} = 0.$$

□

6.6.5 Lemma. *Let $c_1, c_2, c_3, c_4 \in I$ with $c_1 < c_2 < c_3 < c_4$. Then*

$$\sum_{\substack{\text{partitions } (a,b)(a',b') \\ \text{of } \{c_1,c_2,c_3,c_4\}}} \operatorname{sign}(\sigma) D^{a,b}_{i,\bar{i}} D^{a',b'}_{i,\bar{i}} = 0.$$

where $\sigma = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ a & b & a' & b' \end{pmatrix}$, $a < b$, $a' < b'$ *and* $a < a'$.

Proof. Let $c_1, c_2, c_3, c_4 \in I$ with $c_1 < c_2 < c_3 < c_4$. The possible partitions together with the corresponding permutations and signs are

Partition	Permutation	Sign
$(c_1, c_2)(c_3, c_4)$	$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$	1
$(c_1, c_3)(c_2, c_4)$	$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_3 & c_2 & c_4 \end{pmatrix}$	-1
$(c_1, c_4)(c_2, c_3)$	$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_4 & c_2 & c_3 \end{pmatrix}$	1

Therefore

$$\sum_{\substack{\text{partitions } (a,b)(a',b') \\ \text{of } \{c_1,c_2,c_3,c_4\}}} \operatorname{sign}(\sigma) D^{a,b}_{i,\bar{i}} D^{a',b'}_{i,\bar{i}} = D^{c_1,c_2}_{i,\bar{i}} D^{c_3,c_4}_{i,\bar{i}} - D^{c_1,c_3}_{i,\bar{i}} D^{c_2,c_4}_{i,\bar{i}} + D^{c_1,c_4}_{i,\bar{i}} D^{c_2,c_3}_{i,\bar{i}}.$$

Denote $d_{ak} d_{bl}$ by $P^{a,b}_{k,l}$. Then

$$D^{a,b}_{k,l} = P^{a,b}_{k,l} - P^{a,b}_{l,k}.$$

Using this notation we have

$$\begin{aligned}
&\sum_{\substack{\text{partitions } (a,b)(a',b') \\ \text{of } \{c_1,c_2,c_3,c_4\}}} \operatorname{sign}(\sigma) D^{a,b}_{i,\bar{i}} D^{a',b'}_{i,\bar{i}} \\
&= \left\{ \begin{array}{l} (P^{c_1c_2}_{i,\bar{i}} - P^{c_1,c_2}_{\bar{i},i})(P^{c_3c_4}_{i,\bar{i}} - P^{c_3,c_4}_{\bar{i},i}) - (P^{c_1c_3}_{i,\bar{i}} - P^{c_1,c_3}_{\bar{i},i})(P^{c_2c_4}_{i,\bar{i}} - P^{c_2,c_4}_{\bar{i},i}) \\ +(P^{c_1c_4}_{i,\bar{i}} - P^{c_1,c_4}_{\bar{i},i})(P^{c_2c_3}_{i,\bar{i}} - P^{c_2,c_3}_{\bar{i},i}) \end{array} \right\} \\
&= \left\{ \begin{array}{l} P^{c_1c_3}_{i,i} P^{c_2c_4}_{\bar{i},\bar{i}} - P^{c_1c_4}_{i,i} P^{c_2c_3}_{\bar{i},\bar{i}} - P^{c_2c_3}_{i,i} P^{c_1c_4}_{\bar{i},\bar{i}} + P^{c_2c_4}_{i,i} P^{c_1c_3}_{\bar{i},\bar{i}} \\ -P^{c_1c_2}_{i,i} P^{c_3c_4}_{\bar{i},\bar{i}} + P^{c_1c_4}_{i,i} P^{c_2c_3}_{\bar{i},\bar{i}} + P^{c_2c_3}_{i,i} P^{c_1c_4}_{\bar{i},\bar{i}} - P^{c_3c_4}_{i,i} P^{c_1c_2}_{\bar{i},\bar{i}} \\ +P^{c_1c_2}_{i,i} P^{c_3c_4}_{\bar{i},\bar{i}} - P^{c_1c_3}_{i,i} P^{c_2c_4}_{\bar{i},\bar{i}} - P^{c_2c_4}_{i,i} P^{c_1c_3}_{\bar{i},\bar{i}} + P^{c_3c_4}_{i,i} P^{c_1c_2}_{\bar{i},\bar{i}} \end{array} \right\}
\end{aligned}$$

These cancel in pairs to give zero.

$\square$

6.6.6 Definition. Let ( ) denote the empty partition. A ( )-diagram is the empty diagram, as it has no squares. Define the bideterminant of the empty ( )-tableau to be $1 \in A_{\mathbf{k}}^{sp}(\overline{n})$.

6.6.7 Lemma. *Let $t \in \mathbb{N}$ such that $2t \leq n$. Let T be the empty tableau. Then*

$$\omega_t^+(D_T) = \omega_t^+(1_{A_{\mathbf{k}}^{sp}(\overline{n})}) = 0.$$

6.6.8 Example. Let $n = 4$ and $t = 2$. Then

$$\omega_t^+(1_{A_{\mathbf{k}}^{sp}(\overline{n})}) = D_{\begin{array}{|c|}\hline 1\\ \hline \overline{1}\\ \hline 2\\ \hline \overline{2}\\ \hline\end{array}} + D_{\begin{array}{|c|}\hline 1\\ \hline \overline{1}\\ \hline 3\\ \hline \overline{3}\\ \hline\end{array}} + D_{\begin{array}{|c|}\hline 1\\ \hline \overline{1}\\ \hline 4\\ \hline \overline{4}\\ \hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\ \hline \overline{2}\\ \hline 3\\ \hline \overline{3}\\ \hline\end{array}} + D_{\begin{array}{|c|}\hline 2\\ \hline \overline{2}\\ \hline 4\\ \hline \overline{4}\\ \hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\ \hline \overline{3}\\ \hline 4\\ \hline \overline{4}\\ \hline\end{array}} = 0.$$

Proof of Lemma 6.6.7. Let T be the empty tableau and let $t \in \mathbb{N}$ such that $2t \leq n$. Then

$$\omega_t^+(D_T) = \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} D_{\begin{array}{|c|}\hline i_1\\ \hline \overline{i_1}\\ \hline \vdots\\ \hline i_t\\ \hline \overline{i_t}\\ \hline\end{array}}$$

$$= \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} D^{1,2,\ldots,2t-1,2t}_{i_1,\overline{i_1},\ldots,i_t,\overline{i_t}}.$$

Whenever two different rows are equal in a matrix the determinant is zero. Hence the above expansion is equal to

$$\omega_t^+(D_T) = \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 \leq \ldots \leq i_t}} D^{1,2,\ldots,2t-1,2t}_{i_1,\overline{i_1},\ldots,i_t,\overline{i_t}}.$$

By repeated use of the Laplace expansion of determinants, we obtain

$$\omega_t^+(D_T) = \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 \leq \ldots \leq i_t}} \sum_{\substack{(a_1,b_1),\ldots,(a_t,b_t) \\ a_1 < b_1,\ldots,a_t < b_t \\ \{a_1,\ldots,a_t,b_1,\ldots,b_t\} = \{1,\ldots,2t\}}} \operatorname{sign}(\rho) D^{a_1,b_1}_{i_1,\overline{i_1}} \ldots D^{a_t,b_t}_{i_t,\overline{i_t}}$$

where $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2t-1 & 2t \\ a_1 & b_1 & a_2 & b_2 & \dots & a_t & b_t \end{pmatrix}$. If we require that the choice of pairs (a_i, b_i) in the above sum also satisfy $a_1 < \dots < a_t$ then it becomes a sum over partitions of $\{1, 2, \dots, 2t\}$ and we can rewrite the expression as

$$\omega_t^+(D_T) = \sum_{\substack{i_1,\dots,i_t \in I \\ i_1 \le \dots \le i_t}} \sum_{\substack{\text{partitions of } \{1,\dots,2t\} \\ (a_1,b_1),\dots,(a_t,b_t) \\ a_1 < \dots < a_t}} \sum_{\sigma \in S_t} \mathrm{sign}(\rho_\sigma) D_{i_1,\overline{i_1}}^{a_{\sigma(1)},b_{\sigma(1)}} \dots D_{i_t,\overline{i_t}}^{a_{\sigma(t)},b_{\sigma(t)}}$$

where $\rho_\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2t-1 & 2t \\ a_{\sigma(1)} & b_{\sigma(1)} & a_{\sigma(2)} & b_{\sigma(2)} & \dots & a_{\sigma(t)} & b_{\sigma(t)} \end{pmatrix}$. However $\mathrm{sign}(\rho) = \mathrm{sign}(\rho_\sigma)$ where $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2t-1 & 2t \\ a_1 & b_1 & a_2 & b_2 & \dots & a_t & b_t \end{pmatrix}$ because permutations of adjacent pairs are even. Also, the choice of $i_1, \dots, i_t \in I$ is independent of the choice of partition, so we can rewrite the expansion as

$$\omega_t^+(D_T) = \sum_{\substack{\text{partitions of } \{1,\dots,2t\} \\ (a_1,b_1),\dots,(a_t,b_t) \\ a_1 < \dots < a_t}} \mathrm{sign}(\rho) \sum_{\substack{i_1,\dots,i_t \in I \\ i_1 \le \dots \le i_t}} \sum_{\sigma \in S_t} D_{i_1,\overline{i_1}}^{a_{\sigma(1)},b_{\sigma(1)}} \dots D_{i_t,\overline{i_t}}^{a_{\sigma(t)},b_{\sigma(t)}}.$$

We can rearrange the order of the products to give

$$\omega_t^+(D_T) = \sum_{\substack{\text{partitions of } \{1,\dots,2t\} \\ (a_1,b_1),\dots,(a_t,b_t) \\ a_1 < \dots < a_t}} \mathrm{sign}(\rho) \sum_{\substack{i_1,\dots,i_t \in I \\ i_1 \le \dots \le i_t}} \sum_{\sigma \in S_t} D_{i_{\sigma(1)},\overline{i_{\sigma(1)}}}^{a_1,b_1} \dots D_{i_{\sigma(t)},\overline{i_{\sigma(t)}}}^{a_t,b_t}.$$

For a given partition $(a_1, b_1), \dots, (a_t, b_t)$ of $\{1, \dots, 2t\}$ let P denote the polynomial expression

$$P = \sum_{\substack{i_1,\dots,i_t \in I \\ i_1 \le \dots \le i_t}} \sum_{\sigma \in S_t} D_{i_{\sigma(1)},\overline{i_{\sigma(1)}}}^{a_1,b_1} \dots D_{i_{\sigma(t)},\overline{i_{\sigma(t)}}}^{a_t,b_t}.$$

Let

$$P' = P - \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \dots \sum_{i_t=1}^{n} D_{i_1,\overline{i_1}}^{a_1,b_1} \dots D_{i_t,\overline{i_t}}^{a_t,b_t}.$$

By Lemma 6.6.4, $P = P'$.

We are interested in which polynomials appear in P'. Let $j_1, j_2, \dots, j_t \in I$. Then

$$D_{j_1,\overline{j_1}}^{a_1,b_1} D_{j_2,\overline{j_2}}^{a_2,b_2} \dots D_{j_t,\overline{j_t}}^{a_t,b_t}$$

appears exactly once in

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \dots \sum_{i_t=1}^{n} D_{i_1,\overline{i_1}}^{a_1,b_1} D_{i_2,\overline{i_2}}^{a_2,b_2} \dots D_{i_t,\overline{i_t}}^{a_t,b_t},$$

namely when $i_1 = j_1$, $i_2 = j_2$, $\dots$, $i_t = j_t$.

Suppose $j_1, \dots, j_t \in I$ are distinct. Then there is a unique permutation $\gamma \in S_t$ such that

$$j_{\gamma(1)} < j_{\gamma(2)} < \dots < j_{\gamma(t)}.$$

Then

$$D^{a_1,b_1}_{j_1,\overline{j_1}} D^{a_1,b_1}_{j_1,\overline{j_1}} \dots D^{a_t,b_t}_{j_t,\overline{j_t}}$$

appears exactly once in P, that is when $i_1 = j_{\gamma(1)}, i_2 = j_{\gamma(2)}, \dots, i_t = j_{\gamma(t)}$ and $\sigma = \gamma^{-1}$. Hence $D^{a_1,b_1}_{j_1,\overline{j_1}} D^{a_1,b_1}_{j_2,\overline{j_2}} \dots D^{a_t,b_t}_{j_t,\overline{j_t}}$ does not appear in P' as long as $j_1, \dots, j_t$ are distinct.

Suppose $j_1, \dots, j_t \in I$ are not all distinct. Let $\gamma \in S_t$ be a permutation putting the j_i in the order

$$j_{\gamma(1)} \le j_{\gamma(2)} \le \dots \le j_{\gamma(t)}.$$

Let q be the number of distinct elements of I among $j_1, \dots, j_t$. Let $s_1, \dots, s_q$ satisfy

$$\begin{aligned} j_{\gamma(1)} = \dots = j_{\gamma(s_1)} < j_{\gamma(s_1+1)} = \dots = j_{\gamma(s_1+s_2)} < \dots \\ \dots < j_{\gamma(s_1+\dots+s_{q-1}+1)} = \dots = j_{\gamma(s_1+\dots+s_q)} \end{aligned}$$

where $s_1 + \dots + s_q = t$.

Let $S(j_1, \dots, j_t) \subset S_t$ be the subgroup of permutations σ such that

$$(j_{\sigma(1)}, \dots, j_{\sigma(t)}) = (j_1, \dots, j_t).$$

Then $|S(j_1, \dots, j_t)| = s_1!s_2!\dots s_q!$. Let CR be a right transversal of $S(j_1, \dots, j_t)$ in S_t, that is, a set of right coset representatives. By writing P in the form

$$P = \sum_{\substack{i_1,\dots,i_t \in I \\ i_1 \le \dots \le i_t}} \sum_{\sigma_1 \in S(j_1,\dots,j_t)} \sum_{\sigma_2 \in CR} D^{a_1,b_1}_{i_{\sigma_1\sigma_2(1)},\overline{i_{\sigma_1\sigma_2(1)}}} \dots D^{a_t,b_t}_{i_{\sigma_1\sigma_2(t)},\overline{i_{\sigma_1\sigma_2(t)}}}$$

we can see that the term $D^{a_1,b_1}_{j_1,\overline{j_1}} \dots D^{a_t,b_t}_{j_t,\overline{j_t}}$ occurs in P with coefficient $s_1!s_2!\dots s_q!$ and hence occurs in P' with coefficient $(s_1!s_2!\dots s_q! - 1) \in \mathbb{Z}$. Denote $(s_1!s_2!\dots s_q! - 1)$ by $k(j_1, \dots, j_t)$. This coefficient will be the same for any choice $j'_1, \dots, j'_t$ which is a permutation of $j_1, \dots, j_t$. So P' can be expressed as

$$P' = \sum_{\substack{i_1,\dots,i_t \in I \\ i_1 \le \dots \le i_t \\ \text{not all distinct}}} k(i_1, \dots, i_t) \sum_{\substack{\text{distinct rearrangements} \\ j_1,\dots,j_t \in I \\ \text{of } i_1,\dots,i_t}} D^{a_1,b_1}_{j_1,\overline{j_1}} \dots D^{a_t,b_t}_{j_t,\overline{j_t}}.$$

As $P = P'$ then

$$\omega_t^+(D_T) = \sum_{\substack{\text{partitions of } \{1,\dots,2t\} \\ (a_1,b_1),\dots,(a_t,b_t) \\ a_1 < \dots < a_t}} \operatorname{sign}(\rho) \sum_{\substack{i_1,\dots,i_t \in I \\ i_1 \le \dots \le i_t \\ \text{not all distinct}}} k(i_1, \dots, i_t) \sum_{\substack{\text{distinct} \\ \text{rearrangements} \\ j_1,\dots,j_t \in I \\ \text{of } i_1,\dots,i_t}} D^{a_1,b_1}_{j_1,\overline{j_1}} \dots D^{a_t,b_t}_{j_t,\overline{j_t}}.$$

Fix $i_1,\ldots,i_t \in I$, not all distinct. To prove the lemma it is sufficient to show the following expression is zero.

$$Q = \sum_{\substack{\text{partitions of } \{1,\ldots,2t\} \\ (a_1,b_1),\ldots,(a_t,b_t) \\ a_1<\ldots<a_t}} \operatorname{sign}(\rho) \sum_{\substack{\text{distinct rearrangements} \\ j_1,\ldots,j_t \in I \\ \text{of } i_1,\ldots,i_t}} D^{a_1,b_1}_{j_1,\overline{j_1}} \ldots D^{a_t,b_t}_{j_t,\overline{j_t}}$$

where $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2t-1 & 2t \\ a_1 & b_1 & a_2 & b_2 & \ldots & a_t & b_t \end{pmatrix}$. Since the inner sum is over all rearrangements of $i_1,\ldots,i_t$, without repeats, and $i_1,\ldots,i_t$ are not all distinct, then we can assume, without loss of generality, that $i_{t-1} = i_t$ (although we no longer have $i_1 \leq \ldots \leq i_t$).

Fix $a_1,\ldots,a_{t-2},b_1,\ldots,b_{t-2} \in \{1,\ldots,2t\}$, all distinct, such that $a_1 < \ldots < a_{t-2}$ and $a_\mu < b_\mu$ for all $\mu = 1,\ldots,t-2$. Let $j_1,\ldots,j_{t-2}$ be a rearrangement of $i_1,\ldots,i_{t-2}$. We are interested in the products in Q which involve

$$D^{a_1,b_1}_{j_1,\overline{j_1}} \ldots D^{a_{t-2},b_{t-2}}_{j_{t-2},\overline{j_{t-2}}}.$$

Let $\{c_1,c_2,c_3,c_4\} = \{1,\ldots,2t\} \setminus \{a_1,\ldots,a_{t-2},b_1,\ldots,b_{t-2}\}$, with $c_1 < c_2 < c_3 < c_4$. Any partition $(a,b),(a',b')$ of $\{c_1,c_2,c_3,c_4\}$ determines a unique partition of $\{1,\ldots,2t\}$ where (a,b) and (a',b') are put in order amongst (a_1,b_1), (a_2,b_2), $\ldots$, (a_{t-2},b_{t-2}). So we have the partition

$$(a_1,b_1),\ldots,(a_{p_1},b_{p_1}),(a,b),(a_{p_1+1},b_{p_1+1}),\ldots,$$
$$(a_{p_2},b_{p_2}),(a',b'),(a_{p_2+1},b_{p_2+1}),\ldots,(a_{t-2},b_{t-2})$$

where $a_1 < \ldots < a_{p_1} < a < a_{p_1+1} < \ldots < a_{p_2} < a' < a_{p_2+1} < \ldots < a_{t-2}$. Let this partition be $(\alpha_1,\beta_1),\ldots,(\alpha_t,\beta_t)$.

There is a unique rearrangement $k_1,\ldots,k_t$ of $i_1,\ldots,i_t$ such that

$$D^{\alpha_1,\beta_1}_{k_1,\overline{k_1}} \ldots D^{\alpha_t,\beta_t}_{k_t,\overline{k_t}} = D^{a_1,b_1}_{j_1,\overline{j_1}} \ldots D^{a_{t-2},b_{t-2}}_{j_{t-2},\overline{j_{t-2}}} D^{a,b}_{i_{t-1},\overline{i_{t-1}}} \underline{\quad} D^{a',b'}_{i_t,\overline{i_t}}.$$

For each partition of $\{c_1,c_2,c_3,c_4\}$ there is a term in Q involving the product

$$D^{a_1,b_1}_{j_1,\overline{j_1}} \ldots D^{a_{t-2},b_{t-2}}_{j_{t-2},\overline{j_{t-2}}},$$

and all terms involving this product occur in this way. Since permutations which interchange pairs are even, the sign of

$$\begin{pmatrix} 1 & 2 & \ldots & 2t-1 & 2t \\ \alpha_1 & \beta_1 & \ldots & \alpha_t & \beta_t \end{pmatrix}$$

equals the sign of

$$\begin{pmatrix} 1 & 2 & \ldots & 2t-5 & 2t-4 & 2t-3 & 2t-2 & 2t-1 & 2t \\ a_1 & b_1 & \ldots & a_{t-2} & b_{t-2} & a & b & a' & b' \end{pmatrix}.$$

The latter permutation equals

$$\begin{pmatrix} 1 & 2 & \dots & 2t-5 & 2t-4 & 2t-3 & 2t-2 & 2t-1 & 2t \\ a_1 & b_1 & \dots & a_{t-2} & b_{t-2} & c_1 & c_2 & c_3 & c_4 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ a & b & a' & b' \end{pmatrix}.$$

If we collect together the terms involving

$$D^{a_1,b_1}_{j_1,\overline{j_1}} \dots D^{a_{t-2},b_{t-2}}_{j_{t-2},\overline{j_{t-2}}}$$

in Q then we get

$$\pm \left(\sum_{\substack{partitions \\ (a,b),(a',b') \\ of\ \{c_1,c_2,c_3,c_4\}}} \operatorname{sign}(\sigma) D^{a,b}_{i_t,\overline{i_t}} D^{a',b'}_{i_t,\overline{i_t}} \right) D^{a_1,b_1}_{i_1,\overline{i_1}} \dots D^{a_{t-2},b_{t-2}}_{i_{t-2},\overline{i_{t-2}}},$$

remembering that $i_t = i_{t-1}$. By Lemma 6.6.5 this is zero. Thus Q is zero and the lemma is proved.

□

6.6.9 Lemma. *Let $m, t \in \mathbb{N}$ such that $m + 2t \leq n$. Let T be any (1^m)-tableau. Then*

$$\omega_t^+(D_T) = 0.$$

6.6.10 Example. Let $n = 6$, $m = 2$ and $t = 2$. Then

$$\omega_t^+\left(D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline\end{array}}\right) = \left\{\begin{array}{l} D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 2\\\hline \overline{2}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 3\\\hline \overline{3}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 2\\\hline \overline{2}\\\hline 3\\\hline \overline{3}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 2\\\hline \overline{2}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 2\\\hline \overline{2}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} \\ + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 2\\\hline \overline{2}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 3\\\hline \overline{3}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 3\\\hline \overline{3}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 3\\\hline \overline{3}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 4\\\hline \overline{4}\\\hline 5\\\hline \overline{5}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 4\\\hline \overline{4}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 5\\\hline \overline{5}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} \end{array}\right\}$$

$$= D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 2\\\hline \overline{2}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 1\\\hline \overline{1}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 2\\\hline \overline{2}\\\hline 4\\\hline \overline{4}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 2\\\hline \overline{2}\\\hline 6\\\hline \overline{6}\\\hline\end{array}} + D_{\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline 4\\\hline \overline{4}\\\hline 6\\\hline \overline{6}\\\hline\end{array}}$$

$$= 0.$$

Proof of Lemma 6.6.9. Let T be a (1^m)-tableau of the form

$$T = \begin{array}{|c|}\hline s_1\\\hline \vdots\\\hline s_m\\\hline\end{array}$$

where $s_1, \ldots, s_m \in I \cup \overline{I}$. Then

$$\omega_t^+(D_T) = \sum_{\substack{i_1,\ldots,i_t \in I\\ i_1 < \cdots < i_t}} D_{\begin{array}{|c|}\hline s_1\\\hline \vdots\\\hline s_m\\\hline i_1\\\hline \overline{i_1}\\\hline \vdots\\\hline i_t\\\hline \overline{i_t}\\\hline\end{array}}$$

$$= \sum_{\substack{i_1,\ldots,i_t \in I\\ i_1 < \cdots < i_t}} D^{1,\ldots,m,m+1,m+2,\ldots,m+2t-1,m+2t}_{s_1,\ldots,s_m,i_1,\overline{i_1},\ldots,i_t,\overline{i_t}}.$$

Using the Laplace expansion of determinants

$$\omega_t^+(D_T) = \sum_{\substack{(a_1,\ldots,a_m)\\(e_1,\ldots,e_{2t})}} \operatorname{sign}(\rho) \sum_{\substack{i_1,\ldots,i_t\in I\\ i_1<\ldots<i_t}} \left(D^{a_1,\ldots,a_m}_{s_1,\ldots,s_m}\right)\left(D^{e_1,e_2,\ldots,e_{2t-1},e_{2t}}_{i_1,\overline{i_1},\ldots,i_t,\overline{i_t}}\right)$$

where the outer sum is over all partitions of $\{1,\ldots,m+2t\}$ into two sets $\{a_1,\ldots,a_m\}$ and $\{e_1,\ldots,e_{2t}\}$ with $a_1<\ldots<a_m$ and $e_1<\ldots<e_{2t}$, and

$$\rho = \begin{pmatrix} 1 & \ldots & m & m+1 & \ldots & m+2t \\ a_1 & \ldots & a_m & e_1 & \ldots & e_{2t} \end{pmatrix}.$$

This can be rewritten to give

$$\omega_t^+(D_T) = \left(\sum_{\substack{(a_1,\ldots,a_m)\\(e_1,\ldots,e_{2t})}} \operatorname{sign}(\rho) D^{a_1,\ldots,a_m}_{s_1,\ldots,s_m}\right)\left(\sum_{\substack{i_1,\ldots,i_t\in I\\ i_1<\ldots<i_t}} D^{e_1,e_2,\ldots,e_{2t-1},e_{2t}}_{i_1,\overline{i_1},\ldots,i_t,\overline{i_t}}\right).$$

We can adapt the proof of Lemma 6.6.7, in which we show that

$$\sum_{\substack{i_1,\ldots,i_t\in I\\ i_1<\ldots<i_t}} D^{1,2,\ldots,2t-1,2t}_{i_1,\overline{i_1},\ldots,i_t,\overline{i_t}} = 0$$

by replacing the numbers $1,\ldots,2t$ by $e_1,\ldots,e_{2t}$, to show that the right hand factor in the above product is zero.

□

6.6.11 Definition. Let $s\in\{1,\ldots,\lambda_1\}$ and $t\in\mathbb{N}$ satisfy $2t+\mu_s\le n$ where μ is the conjugate partition to λ. Define a *$\lambda^{s,t}$-diagram* to be a λ-diagram with $2t$ squares added to the bottom of the s^{th} column. A *$\lambda^{s,t}$-tableau* is a $\lambda^{s,t}$-diagram with entries from $I\cup\overline{I}$. Let $D^{sp}_{\lambda^{s,t},\mathbf{k}}$ be the **k**-span of all bideterminants D_T as T ranges over all $\lambda^{s,t}$-tableaux.

Recall that if T_1, …, T_{λ_1} denote the columns of a λ-tableau T then

$$D_T = D_{T_1}\ldots D_{T_{\lambda_1}}.$$

6.6.12 Definition. Let $s\in\{1,\ldots,\lambda_1+1\}$ and $t\in\mathbb{N}$ satisfy $2t+\mu_s\le n$. Define the *expansion operator* $E_{s,t}: D^{sp}_{\lambda,\mathbf{k}}\to D^{sp}_{\lambda^{s,t},\mathbf{k}}$ to be the linear extension of

$$E_{s,t}(D_T) = \begin{cases} D_{T_1}\ldots D_{T_{s-1}}\omega_t^+(D_{T_s})D_{T_{s+1}}\ldots D_{T_{\lambda_1}} & \text{when } s\ne\lambda_1+1 \\ D_{T_1}\ldots D_{T_{\lambda_1}}\omega_t^+(1_{A_{\mathbf{k}}^{sp}(\overline{n})}) & \text{when } s=\lambda_1+1 \end{cases}$$

6.6.13 Notation. Let T be a λ-tableau and let $i_1, \ldots, i_t \in I$ for some $t \in \mathbb{N}$ such that $2t + \mu_s \leq n$. For $s \in \{1, \ldots, \lambda_1 + 1\}$ denote by $T^s(i_1, \ldots, i_t)$ the $\lambda^{s,t}$-tableau obtained by adjoining $2t$ squares to the bottom of column s and putting in one entry per square from $i_1, \bar{i}_1, \ldots, i_t, \bar{i}_t$ in order from top to bottom.

6.6.14 Lemma. *Let $s \in \{1, \ldots, \lambda_1 + 1\}$ and $t \in \mathbb{N}$ such that $2t + \mu_s \leq n$. Let T be a λ-tableau. Then*

$$E_{s,t}(D_T) = \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} D_{T^s(i_1,\ldots,i_t)} = 0$$

Proof. This follows from the definition of $E_{s,t}$ and Lemmas 6.6.7 and 6.6.9.

□

6.6.15 Proposition. *The set of bideterminants D_T, as T ranges over all symplectic λ-tableaux, forms a spanning set for $D^{sp}_{\lambda,\mathbf{k}}$.*

Proof. We briefly recall some definitions from Chapter 5. The modulus function $|\ |$ on $I \cup \bar{I}$ satisfies $|i| = |\bar{i}| = i$ for all $i \in I$. The height of a λ-tableau is the sum of the moduli of its entries. Two λ-tableaux T and T' are equivalent if $\mathrm{ht}(T) = \mathrm{ht}(T')$, and we denote the equivalence class containing T by $[T]$. Define $T \leq T'$ if and only if $\mathrm{ht}(T) \leq \mathrm{ht}(T')$. Then $T \leq T'$ and $T' \leq T$ implies $[T] = [T']$. This gives a total ordering on the set of equivalence classes.

Suppose T is a semistandard non-symplectic λ-tableau. To prove the lemma it is sufficient to show that D_T can be expressed as a linear combination of bideterminants of symplectic λ-tableaux.

Let $f : T \longrightarrow D^{sp}_{\lambda,\mathbf{k}}$ be the function sending a λ-tableau to its bideterminant D_T. By the properties of determinants and Lemma 6.6.14 this satisfies the conditions of Lemma 5.3.14. Hence D_T can be expressed as a linear combination of bideterminants of λ-tableaux T' all satisfying $[T] < [T']$.

Let T' be such a λ-tableau. If T' is not semistandard then, by Carter and Lusztig [1] (39) p. 215, $D_{T'}$ can be expressed as a linear combination of bideterminants of semistandard λ-tableaux T'' which are all permutations of T'. Let T'' be a λ-tableau which occurs in this way. Then $[T''] = [T'] > [T]$.

If T'' is not symplectic we may use Lemma 5.3.14 again to obtain an expression for $D_{T''}$ as a linear combination of bideterminants of λ-tableaux of strictly greater height than T''.

Each time this process is repeated the new tableaux are of strictly greater height than the tableaux they replace. As there are only a finite number of heights this process must terminate, and we are left with an expression for D_T as a linear combination of bideterminants of λ-tableaux which are all symplectic.

□

6.7 The Property of Linear Independence.

This section is an adaptation of a more general proof in the context of the general linear group in Green [2].

Recall the basic λ-tableau T_0 which contains all entries equal to i in the squares in the i^{th} row for all $i \in \{1, \dots, \mu_1\}$, where μ is the conjugate partition to λ.

6.7.1 Definitions. Let $i, j \in \mathbb{N}$ satisfy $1 \leq i < j \leq n+1$. Define various subsets of $\mathcal{T}_{sp}$ where $\mathcal{T}_{sp}$ is the set of all symplectic λ-tableaux with entries from $I \cup \overline{I}$, as follows

$\Gamma_{i,j}$ is the set of $T \in \mathcal{T}_{sp}$ such that T is identical to T_0 in the first $i-1$ rows and the entries in the squares in the i^{th} row of T are either i or not less than j;

$\Gamma_{i,\overline{j}}$ is the set of $T \in \mathcal{T}_{sp}$ such that T is identical to T_0 in the first $i-1$ rows and the entries in the squares in the i^{th} row of T are either i or not less than $\overline{j}$;

$\Gamma_{i,\overline{i}}$ is the set of $T \in \mathcal{T}_{sp}$ such that T is identical to T_0 in the first $i-1$ rows.

Notice that $\Gamma_{1,\overline{1}} = \mathcal{T}_{sp}$ and $\Gamma_{n,n+1} = \{T_0\}$. Also for all $i \in I$ with $i \geq 2$

$$\Gamma_{i,\overline{i}} = \Gamma_{i-1,n+1},$$

for all $i, j \in I$ with $i < j$

$$\Gamma_{i,j} \supset \Gamma_{i,\overline{j}} \supset \Gamma_{i,j+1}$$

and for all $i \in I$

$$\Gamma_{i,\overline{i}} \supset \Gamma_{i,i+1}.$$

Consequently, there is a chain of subsets of $\mathcal{T}$

$$\mathcal{T}_{sp} = \Gamma_{1,\overline{1}} \supset \Gamma_{1,2} \supset \Gamma_{1,\overline{2}} \supset \Gamma_{1,3} \supset \dots \supset \Gamma_{1,n+1} \supset \Gamma_{2,\overline{2}} \supset \Gamma_{2,3} \supset \dots \supset \Gamma_{n,n+1} = \{T_0\}. \quad (\star)$$

6.7.2 Definitions. Let $i \in \{1, \dots, \mu_1\}$ and $j \in I \cup \overline{I}$. Define $\vartheta_{i,j} : \mathcal{T} \to \mathcal{T}$ to be the map which sends a λ-tableau T to the λ-tableau $\vartheta_{i,j}(T)$ obtained from T by replacing all entries j in the i^{th} row by i.

Let $i \in I$ and $j \in I \cup \overline{I}$. Define $N_{i,j} : \mathcal{T} \to \mathbb{Z}$ to be the map sending T to the number of entries j in the i^{th} row of T.

6.7.3 Example. Let $\lambda = (3,2,1)$ and let $T = \begin{array}{|c|c|c|}\hline 1 & \overline{1} & \overline{3} \\ \hline 2 & \overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array}$. Then $N_{2,\overline{3}}(T) = 1$, because there is one entry equal to $\overline{3}$ in row 2, and

$$\vartheta_{2,\overline{3}}(T) = \begin{array}{|c|c|c|}\hline 1 & \overline{1} & \overline{3} \\ \hline 2 & 2 \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array}.$$

Recall from Definition 3.7.2, the functions $\psi_v^{x_{i,j}}$, $\psi_{v,w}^{y_{i,j}}$ and $\psi_v^{z_i}$ from $A_{\mathbf{k}}^{sp}(\overline{n})$ to itself.

6.7.4 Lemma. *Let $i,j \in I$ with $i < j$. Let $v \in \mathbb{N}$ and let $T \in \Gamma_{i,\bar{j}}$. Then*

$$\psi_v^{x_{i,j}}(D_T) = \begin{cases} 0 & \text{if } v > N_{i,\bar{j}}(T) \\ \sum_{T'} D_{T'} & \text{otherwise} \end{cases}$$

where the sum is over all λ-tableaux T' obtained from T by replacing v entries $\bar{j}$ by i in row i.

Proof. Let $T \in \Gamma_{i,\bar{j}}$. The first $i-1$ rows of T do not contain any entries equal to $\bar{i}$ because they are identical to the first $i-1$ rows of the basic λ-tableau T_0. All entries in row i are either i or not less than $\bar{j}$, and so there are no entries $\bar{i}$ in row i. As T is symplectic there are no entries $\bar{i}$ in the rows below the i^{th} row. Therefore T does not contain any entries equal to $\bar{i}$.

Hence by Definition 3.7.2 we have

$$\psi_v^{x_{i,j}}(D_T) = \sum_{T'} D_{T'}$$

where the sum is over all λ-tableaux T' obtained from T by replacing v entries $\bar{j}$ by i.

Consider which rows will contain entries equal to $\bar{j}$. There are no entries $\bar{j}$ in the first $i-1$ rows of T for the same reason that there are no entries $\bar{i}$ in these rows. However, there may be entries equal to $\bar{j}$ in the i^{th} row and in the rows below.

Suppose T' is obtained from T by replacing v entries $\bar{j}$ by i and not all of these replacements are in row i. Let k be a column in which $\bar{j}$ was replaced by i in a row below the i^{th} row. The square in the i^{th} row of column k must contain an entry less than j because T is symplectic, and by the definition of $\Gamma_{i,\bar{j}}$ that entry must be i. Thus column k in T' contains two different squares which have the same entry i. Thus $D_{T'} = 0$, and the lemma is proved.

□

6.7.5 Corollary. *Let $i,j \in I$ with $i < j$ and let $T \in \Gamma_{i,\bar{j}}$. Then*

$$\psi_{N_{i,\bar{j}}(T)}^{x_{i,j}}(D_T) = D_{\vartheta_{i,\bar{j}}(T)}.$$

6.7.6 Lemma. *Let $i,j \in I$ with $i < j$. Let $v,w \in \mathbb{N}$ and let $T \in \Gamma_{i,j}$. Then*

$$\psi_{v,w}^{y_{i,j}}(D_T) = \begin{cases} 0 & \text{if } w \neq 0 \\ 0 & \text{if } v > N_{i,j}(T) \\ \sum_{T'} D_{T'} & \text{otherwise} \end{cases}$$

where the sum is over all λ-tableaux T' obtained from T by replacing v entries j by i in row i.

Proof. Let $T \in \Gamma_{i,j}$. By a similar argument used in the last proof T contains no entries $\bar{i}$. By Definition 3.7.2 $\psi^{y_{i,j}}_{v,w}(D_T) = 0$ if $w \neq 0$.

Let $v \in \mathbb{N}$. Then

$$\psi^{y_{i,j}}_{v,0}(D_T) = \sum_{T'} D_{T'}$$

where the sum is over all λ-tableaux T' obtained from T by replacing v entries j by i.

There are no entries j in the squares above the i^{th} row of T since the first $i-1$ rows of T are identical to the first $i-1$ rows of T_0.

Suppose that T' is obtained from T by replacing v entries j by i, but with not all replacements happening in row i. let k be a column in which j was replaced by i in a row below the i^{th} row. The square in the i^{th} row and k^{th} column of T must contain an entry less than j because T is symplectic, and so which must be i by the definition of $\Gamma_{i,j}$. Therefore T' contains the entry i in two different squares in column k, and so $D_{T'} = 0$. Hence the lemma is proved.

□

6.7.7 Corollary. *Let $i, j \in I$ with $i < j$ and let $T \in \Gamma_{i,j}$. Then*

$$\psi^{y_{i,j}}_{N_{i,j}(T),0}(D_T) = D_{\vartheta_{i,j}(T)}.$$

6.7.8 Lemma. *Let $i \in I$, let $v \in \mathbb{N}$ and let $T \in \Gamma_{i,\bar{i}}$. Then*

$$\psi^{z_i}_v(D_T) = \begin{cases} 0 & \textit{if } v > N_{i,\bar{i}}(T) \\ \sum_{T'} D_{T'} & \textit{otherwise} \end{cases}$$

where the sum is over all λ-tableaux T' obtained from T by replacing v entries $\bar{i}$ by i in row i.

Proof. By Definition 3.7.2 we have

$$\psi^{z_i}_v(D_T) = \sum_{T'} D_{T'}$$

where the sum is over all λ-tableaux T' obtained from T by replacing v entries $\bar{i}$ by i. Since T is identical to T_0 in the first $i-1$ rows, and is symplectic, the only entries equal to $\bar{i}$ in T are in the squares in the i^{th} row. The lemma follows.

□

6.7.9 Corollary. *Let $i \in I$ and let $T \in \Gamma_{i,\bar{i}}$. Then*

$$\psi^{z_i}_{N_{i,\bar{i}}(T)}(D_T) = D_{\vartheta_{i,\bar{i}}(T)}.$$

6.7.10 Example. let $n = 4$, $i = 2$ and $j = 3$. Let $T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & \bar{3} & \bar{3} & 4 \\ \cline{1-4} 3 & 4 & \bar{4} \\ \cline{1-3}\end{array}$. Then $T \in \Gamma_{2,\bar{3}}$ and $N_{2,\bar{3}}(T) = 2$. So $\vartheta_{2,\bar{3}}(T) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 4 \\ \cline{1-4} \bar{3} & 4 & \bar{4} \\ \cline{1-3}\end{array}$ and $\psi_2^{x_{2,3}}(D_T) = D_{\vartheta_{2,\bar{3}}(T)}$.

Let $U = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 3 & 3 & \bar{3} \\ \cline{1-3} 4 & \bar{4} & \bar{4} \\ \cline{1-3}\end{array}$. Then $U \in \Gamma_{2,3}$ and $N_{2,3}(U) = 2$. So $\vartheta_{2,3}(U) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & \bar{3} \\ \cline{1-3} 4 & \bar{4} & \bar{4} \\ \cline{1-3}\end{array}$ and $\psi_{2,0}^{y_{2,3}}(D_U) = D_{\vartheta_{2,3}(U)}$.

Let $S = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & \bar{2} & \bar{2} & \bar{2} & 3 \\ \hline \bar{2} & 3 & 3 \\ \cline{1-3}\end{array}$. Then $S \in \Gamma_{2,\bar{2}}$, $N_{2,\bar{2}}(S) = 3$, $\vartheta_{2,\bar{2}}(S) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 3 \\ \hline \bar{2} & 3 & 3 \\ \cline{1-3}\end{array}$ and $\psi_3^{z_2}(D_S) = D_{\vartheta_{2,\bar{2}}(S)}$.

6.7.11 Definition. A subset $\Gamma \subset \mathcal{T}_{sp}$ is *independent* if the set

$$\{D_T \; ; \; T \in \Gamma\}$$

is linearly independent over $\mathbf{k}$.

6.7.12 Lemma. *let $i, j \in I$ with $i < j$. If $\Gamma_{i,j+1}$ is independent then so is $\Gamma_{i,\bar{j}}$.*

Proof. Assume that $\Gamma_{i,j+1}$ is independent. Let $\Gamma \subset \Gamma_{i,\bar{j}}$ be a non-empty subset and let $\{a_T; T \in \Gamma\}$ be a family of non-zero elements of $\mathbf{k}$. Let

$$a = \sum_{T \in \Gamma} a_T D_T \in D^{sp}_{\lambda,\mathbf{k}}.$$

We wish to show that $a \neq 0$. Assume $a = 0$ for a contradiction. $Sp_{2n}(\mathbf{k})$ acts on the left of $D^{sp}_{\lambda,\mathbf{k}}$. let $t \in \mathbf{k}$ and let $x_{i,j}(t) \in Sp_{2n}(\mathbf{k})$ be given by

$$x_{i,j}(t) = I + t(E_{i,\bar{j}} + E_{j,\bar{i}}).$$

We are interested in the image of a under the action of $x_{i,j}(t)$. By Lemma 3.7.3

$$\begin{aligned} x_{i,j}(t) \circ a &= \sum_{T \in \Gamma} a_T \sum_{v=0}^{r} t^v \psi_v^{x_{i,j}}(D_T) \\ &= \sum_{T \in \Gamma} a_T \sum_{v=0}^{N_{i,\bar{j}}(T)} t^v \psi_v^{x_{i,j}}(D_T) \quad \text{by Lemma 6.7.4} \\ &= a_0 + a_1 t + \ldots + a_N t^N \end{aligned}$$

where $N = \max\{N_{i,\bar{j}}(T); T \in \Gamma\}$ and $a_0, \ldots, a_N \in A_{\mathbf{k}}^{sp}(\overline{n}, r)$ are independent of $t \in \mathbf{k}$. Notice that $a_0 = a = 0$.

Let $\Gamma^* \subset \Gamma$ be the subset of all $T \in \Gamma$ such that $N_{i,\bar{j}}(T) = N$. By Lemma 6.7.4 and Corollary 6.7.5 $a_N \in A_{\mathbf{k}}^{sp}(\overline{n}, r)$ is given by

$$\begin{aligned} a_N &= \sum_{T \in \Gamma^*} a_T \psi_N^{x_{i,j}}(D_T) \\ &= \sum_{T \in \Gamma^*} a_T D_{\vartheta_{i,\bar{j}}(T)}. \end{aligned}$$

Clearly Γ^* is not empty, and for each $T \in \Gamma^*$ we have $\vartheta_{i,\bar{j}}(T) \in \Gamma_{i,j+1}$.

Moreover, we claim that $\vartheta_{i,\bar{j}}(T)$ runs over a certain non-empty subset of $\Gamma_{i,j+1}$ without repetition as T runs over Γ^*. Suppose that there exist $T, T' \in \Gamma^*$ with $T \neq T'$ and $\vartheta_{i,\bar{j}}(T) = \vartheta_{i,\bar{j}}(T')$. Since $N_{i,\bar{j}}(T) = N_{i,\bar{j}}(T') = N$ the same number of replacements are made by the map $\vartheta_{i,\bar{j}}$ in the same row. So T and T' can differ at most only in the order of the entries in the i^{th} row. Since T and T' are *symplectic, they are both non-decreasing* from left to right along rows, and so $T = T'$. This proves the claim.

Hence the sum $\sum_{T \in \Gamma^*} a_T D_{\vartheta_{i,\bar{j}}(T)}$ is a non-trivial linear combination of elements $D_{T'}$ where $T' \in \Gamma_{i,j+1}$. Since $\Gamma_{i,j+1}$ is independent we have proved that $a_N \neq 0$.

We claim, therefore, that $x_{i,j}(t) \circ a \neq 0$ for some $t \in \mathbf{k}$. Suppose not. Then

$$a_1 t + a_2 t^2 + \ldots + a_N t^N = 0$$

for all $t \in \mathbf{k}$. As the field $\mathbf{k}$ is infinite we can choose N distinct non-zero elements $x_1, \ldots, x_N \in \mathbf{k}$ with the condition that $x_i - x_j \neq 0$ for all pairs $i, j \in \{1, \ldots, N\}$ with $i \neq j$. Let

$$X = \begin{pmatrix} x_1^N & x_1^{N-1} & \ldots & x_1 \\ x_2^N & x_2^{N-1} & \ldots & x_2 \\ \vdots & \vdots & & \vdots \\ x_N^N & x_N^{N-1} & \ldots & x_N \end{pmatrix}.$$

Then we have

$$X \begin{pmatrix} a_N \\ a_{N-1} \\ \vdots \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Also det $X = \prod_{i=1}^{N} x_i \prod_{1\le i<j\le N}(x_i - x_j)$ which is non-zero due to the choice of the x_i. So X is invertible and so $a_N = a_{N-1} = \ldots = a_1 = 0$. This contradicts $a_N \neq 0$ and so $x_{i,j}(t) \circ a \neq 0$ for some $t \in \mathbf{k}$. This proves the claim.

However this contradicts the assumption that $a = 0$. Therefore $a \neq 0$ and $\Gamma_{i,\bar{j}}$ is independent.

□

6.7.13 Lemma. *Let $i, j \in I$ with $i < j$. If $\Gamma_{i,\bar{j}}$ is independent then so is $\Gamma_{i,j}$.*

Proof. Assume that $\Gamma_{i,\bar{j}}$ is independent and let $\Gamma \subset \Gamma_{i,j}$ with $\Gamma \neq \emptyset$. Let $\{a_T; T \in \Gamma\}$ be a family of non-zero elements of $\mathbf{k}$. Let

$$a = \sum_{T\in\Gamma} a_T D_T.$$

For a contradiction, assume that $a = 0$.

Let $t \in \mathbf{k}$ and let $y_{i,j}(t) \in Sp_{2n}(\mathbf{k})$ be the element defined in Chapter 3 by

$$y_{i,j}(t) = I + t(E_{i,j} - E_{\bar{i},\bar{j}}).$$

By Lemma 3.7.4

$$y_{i,j}(t) \circ a = \sum_{T\in\Gamma} a_T \sum_{v=0}^{r} \sum_{w=0}^{r} (-1)^w t^{v+w} \psi_{v,w}^{y_{i,j}}(D_T).$$

By Lemma 6.7.6 this gives

$$\begin{aligned} y_{i,j}(t) \circ a &= \sum_{T\in\Gamma} a_T \sum_{v=0}^{N_{i,j}(T)} t^v \psi_{v,0}^{y_{i,j}}(D_T). \\ &= a_0 + a_1 t + a_2 t^2 + \ldots + a_N t^N \end{aligned}$$

where $N = \max\{N_{i,j}(T); T \in \Gamma\}$ and $a_0, \ldots, a_N \in A_{\mathbf{k}}^{sp}(\overline{n}, r)$ are independent of $t \in \mathbf{k}$. Notice that $a_0 = a = 0$.

Let $\Gamma^* \subset \Gamma$ be the subset of tableaux T such that $N_{i,j}(T) = N$. By Lemma 6.7.6 and Corollary 6.7.7 the coefficient a_N is given by

$$\begin{aligned} a_N &= \sum_{T\in\Gamma^*} a_T \psi_{N,0}^{y_{i,j}}(D_T) \\ &= \sum_{T\in\Gamma^*} a_T D_{\vartheta_{i,j}(T)}. \end{aligned}$$

Clearly Γ^* is not empty and for each $T \in \Gamma^*$ we have $\vartheta_{i,j}(T) \in \Gamma_{i,\bar{j}}$. Moreover, $\vartheta_{i,j}(T)$ runs over a certain non-empty subset of $\Gamma' \subset \Gamma_{i,\bar{j}}$ without repetition as T runs over Γ^*. Hence

$$\sum_{a \in \Gamma^*} a_T D_{\vartheta_{i,j}(T)}$$

is a non-trivial linear combination of elements $D_{T'}$ where $T' \in \Gamma' \subset \Gamma_{i,\bar{j}}$. Since $\Gamma_{i,\bar{j}}$ is independent we know that $a_N \neq 0$. Since **k** is infinite and $a_N \neq 0$ there is some $t \in \mathbf{k}$ such that

$$y_{i,j}(t) \circ a = a_1 t + \ldots + a_N t^N \neq 0.$$

This contradicts the assumption that $a = 0$ and so the lemma is proved.

□

6.7.14 Lemma. *Let $i \in I$. If $\Gamma_{i,i+1}$ is independent then so is $\Gamma_{i,\bar{i}}$.*

Proof. Assume that $\Gamma_{i,i+1}$ is independent and let $\Gamma \subset \Gamma_{i,\bar{i}}$ with $\Gamma \neq \emptyset$. Let $\{a_T; T \in \Gamma\}$ be a family of non-zero elements of **k**. Let

$$a = \sum_{T \in \Gamma} a_T D_T.$$

Let $t \in \mathbf{k}$ and let $z_i(t) \in Sp_{2n}(\mathbf{k})$ be the element defined by

$$z_i(t) = I + tE_{i,\bar{i}}.$$

Then Lemma 3.7.5 gives

$$\begin{aligned} z_i(t) \circ a &= \sum_{T \in \Gamma} a_T \sum_{v=0}^{r} t^v \psi_v^{z_i}(D_T) \\ &= \sum_{T \in \Gamma} a_T \sum_{v=0}^{N_{i,\bar{i}}(T)} t^v \psi_v^{z_i}(D_T) \text{ by Lemma 6.7.8} \\ &= a_0 + a_1 t + a_2 t^2 + \ldots + a_N t^N, \end{aligned}$$

where $a_0, \ldots, a_N \in A_{\mathbf{k}}^{sp}(\overline{n}, r)$ are independent of $t \in \mathbf{k}$ and $N = \max\{N_{i,\bar{i}}(T); T \in \Gamma\}$.

Assume that $a = 0$, for a contradiction. Notice that $a_0 = a = 0$.

Let $\Gamma^* \subset \Gamma$ be the subset of tableaux T satisfying $N_{i,\bar{i}}(T) = N$. By Lemma 6.7.8 and Corollary 6.7.9

$$a_N = \sum_{T \in \Gamma^*} a_T D_{\vartheta_{i,\bar{i}}(T)}.$$

This is a non-empty linear combination of the elements $D_{\vartheta_{i,\bar{i}}(T)}$ with $T \in \Gamma^*$.

As T runs over Γ^* the tableau $\vartheta_{i,\bar{i}}(T)$ runs over a non-empty subset $\Gamma' \subset \Gamma_{i,i+1}$ without repetitions. So the above expression for a_N is a non-empty linear combination of elements $D_{T'}$ where $T' \in \Gamma' \subset \Gamma_{i,i+1}$. Since $\Gamma_{i,i+1}$ is independent $a_N \neq 0$. Since $\mathbf{k}$ is infinite there must be a $t \in \mathbf{k}$ such that

$$a_1 t + a_2 t^2 + \ldots + a_N t^N \neq 0$$

and so $z_i(t) \circ a \neq 0$. This contradicts the assumption that $a = 0$. So the lemma is proved.

□

6.7.15 Proposition. *The set*

$$\{D_T \ ; \ T \text{ is a symplectic } \lambda\text{-tableau } \}$$

is linearly independent over $\mathbf{k}$.

Proof. Let $I_{2n} \in Sp_{2n}(\mathbf{k})$ be the identity matrix and let T_0 denote the basic λ-tableau. Then

$$D_{T_0}(I_{2n}) = 1$$

and so $D_{T_0} \not\equiv 0$. Hence $\{D_{T_0}\}$ is linearly independent over $\mathbf{k}$, that is, $\Gamma_{n,n+1}$ is independent.

By induction on the chain of subsets $(\star)$ using Lemmas 6.7.12, 6.7.13 and 6.7.14 we have that $\Gamma_{1,\bar{1}}$, which is the set of all symplectic λ-tableaux, is independent. So the lemma is proved.

□

6.7.16 Theorem. *The set*

$$\{D_T \ ; \ T \text{ is a symplectic } \lambda\text{-tableau } \}$$

is a basis for $D^{sp}_{\lambda,\mathbf{k}}$.

Proof. By Proposition 6.6.15 this set is a spanning set and, by Proposition 6.7.15, it is linearly independent.

□

6.8 The Equality of $I^{sp}_{\lambda,\mathbf{k}}$ and $D^{sp}_{\lambda,\mathbf{k}}$.

We recall some earlier notation. The subalgebra $A_{\mathbf{k}}^{sp}(\overline{n}, r) \subset A_{\mathbf{k}}^{sp}(\overline{n})$ consists of all polynomials in the coefficient functions $d_{i,j}$ which are homogeneous of degree r. We denote by $I(n,r)$ and $I(\overline{n},r)$ the sets of r-tuples with entries from $\{1,..,n\}$ and $\{1,\ldots,n,\overline{1},\ldots,\overline{n}\}$ respectively. Finally, for $\alpha, \beta \in I(\overline{n}, r)$, we set

$$d_{\alpha,\beta} = d_{\alpha_1,\beta_1} d_{\alpha_2,\beta_2} \ldots d_{\alpha_r,\beta_r} \in A_{\mathbf{k}}^{sp}(\overline{n}, r)$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$.

We begin by proving some lemmas involving the symplectic functions defined in Chapter 3.

6.8.1 Lemma. *Let $\alpha \in I(n,r)$ and let $\beta \in I(\overline{n},r)$. Let $i,j \in I$ satisfy $i < j$ and let $v \in \mathbb{N}$. Then*

$$\psi_v^{a_{i,j}}(d_{\alpha,\beta}) = 0.$$

Notice that α contains no barred elements.

Proof. Let α, β, i, j, v be as above. Then

$$\psi_v^{a_{i,j}}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\gamma,\beta}$$

where the sum is over all $\gamma \in I(\overline{n}, r)$ obtained from α by replacing some entries $\overline{i}$ by j and some entries $\overline{j}$ by i with a total of v replacements. Since $\alpha \in I(n,r)$ contains no barred elements there are no entries in α equal to $\overline{i}$ or $\overline{j}$. Thus the sum is empty and its value is zero.

□

6.8.2 Lemma. *Let $\alpha \in I(n,r)$ and let $\beta \in I(\overline{n},r)$. Let $i,j \in I$ satisfy $i < j$ and let $v, w \in \mathbb{Z}$ such that $v > 0$ and $w \geq 0$. Then*

$$\psi_{v,w}^{b_{i,j}}(d_{\alpha,\beta}) = 0.$$

Again note that α contains no barred elements.

Proof. Let $\alpha, \beta, i, j, v, w$ be as above. Then

$$\psi_{v,w}^{b_{i,j}}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\gamma,\beta}$$

where the sum is over all $\gamma \in I(\overline{n}, r)$ obtained from α by replacing v entries $\overline{i}$ by $\overline{j}$ and w entries j by i. Since $\alpha \in I(n,r)$ contains no barred elements there are no entries in α equal to $\overline{i}$. Thus the sum is empty and its value is zero.

□

6.8.3 Lemma. *Let $\alpha \in I(n,r)$ and let $\beta \in I(\overline{n},r)$. Let $i \in I$ and let $v \in \mathbb{N}$. Then*

$$\psi_v^{c_i}(d_{\alpha,\beta}) = 0.$$

Again, notice that α contains no barred elements.

Proof. Let α, β, i, v be as above. Then

$$\psi_v^{c_i}(d_{\alpha,\beta}) = \sum_{\gamma} d_{\gamma,\beta}$$

where the sum is over all $\gamma \in I(\overline{n},r)$ obtained from α by replacing v entries $\overline{i}$ by i. Since $\alpha \in I(n,r)$ contains no barred elements there are no entries in α equal to $\overline{i}$. Thus the sum is empty and its value is zero.

□

We now make use of the symplectic relations defined in Definition 3.6.3.

6.8.4 Lemma. *Let $f \in D^{sp}_{\lambda,\mathbf{k}}$. Then f satisfies the symplectic relations.*

Proof. Since $D^{sp}_{\lambda,\mathbf{k}}$ is the $\mathbf{k}$-span of the bideterminants of all λ-tableaux it will be enough to show that D_T satisfies the symplectic relations for any λ-tableau T.

Let T be a λ-tableau with columns $T_1, \ldots, T_{\lambda_1}$. Then

$$D_T = D_{T_1} D_{T_2} \ldots D_{T_{\lambda_1}}$$

and if $D_{T_1}, \ldots, D_{T_{\lambda_1}}$ satisfy the symplectic relations so does D_T.

Let $m \in \mathbb{N}$ with $m \leq n$. It is sufficient to prove that D_T satisfies the symplectic relations whenever T is a (1^m)-tableau. Let T be such a tableau. So

$$T = \begin{array}{|c|} \hline t_1 \\ \hline \vdots \\ \hline t_m \\ \hline \end{array}$$

for some $t_1, t_2, \ldots, t_m \in I \cup \overline{I}$. Then

$$D_T = \begin{vmatrix} d_{1,t_1} & d_{1,t_2} & \ldots & d_{1,t_m} \\ d_{2,t_1} & d_{2,t_2} & \ldots & d_{2,t_m} \\ \vdots & \vdots & & \vdots \\ d_{m,t_1} & d_{m,t_2} & \cdots & d_{m,t_m} \end{vmatrix}$$
$$= \sum_{\sigma \in S_m} \operatorname{sign}(\sigma) d_{1,t_{\sigma(1)}} d_{2,t_{\sigma(2)}} \ldots d_{m,t_{\sigma(m)}}.$$

So D_T is a linear combination of monomials $d_{\alpha,\beta}$ where $\alpha = (1,2,\dots,m) \in I(n,r)$ and $\beta = (t_{\sigma(1)},\dots,t_{\sigma(m)}) \in I(\overline{n},r)$ for some $\sigma \in S_m$.

By Lemmas 6.8.1, 6.8.2 and 6.8.3, whenever α and β are of this form

$$\psi_v^{a_{i,j}}(d_{\alpha,\beta}) = \psi_{v,w}^{b_{i,j}}(d_{\alpha,\beta}) = 0$$

for all $i,j \in I$ with $i < j$, all $v \in \mathbb{N}$ and all $w \in \mathbb{Z}$ such that $w \geq 0$. Also

$$\psi_v^{c_i}(d_{\alpha,\beta}) = 0$$

for all $i \in I$ and $v \in \mathbb{N}$.

Let $i,j \in I$ with $i < j$ and let $w \in \mathbb{N}$. It only remains to find the value of $\psi_{0,w}^{b_{i,j}}(d_{\alpha,\beta})$. Let $\alpha = (1,2,\dots,m)$ and let $\beta \in I(\bar{n},r)$. Then

$$\psi_{0,w}^{b_{i,j}}(d_{\alpha,\beta}) = (-1)^w \sum_{\gamma} d_{\gamma,\beta}$$

where the sum is over all $\gamma \in I(\overline{n},r)$ obtained from α by replacing w entries j by i.

If $m < j$ then α contains no entries equal to j and the sum is zero. If $m \geq j$ then α contains one entry equal to j. In this case if $w > 1$ then the sum is again zero. Hence

$$\psi_{0,w}^{b_{i,j}}(d_{\alpha,\beta}) = 0$$

unless $w = 1$ and $m \geq j$.

So assume that $w = 1$ and $m \geq j$. Then

$$\psi_{0,1}^{b_{i,j}}(d_{\alpha,\beta}) = -d_{\alpha',\beta}$$

where α' is obtained from α by replacing j by i. Hence

$$\psi_{0,1}^{b_{i,j}}(D_T) = \begin{vmatrix} d_{1,t_1} & & d_{1,t_i} & & d_{1,t_j} & & d_{1,t_m} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ d_{i,t_1} & \cdots & d_{i,t_i} & \cdots & d_{i,t_j} & \cdots & d_{i,t_m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ d_{i,t_1} & \cdots & d_{i,t_i} & \cdots & d_{i,t_j} & \cdots & d_{i,t_m} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ d_{m,i_1} & & d_{m,i_i} & & d_{m,i_j} & & d_{m,i_m} \end{vmatrix}$$

and this is zero since there are two equal rows in the determinant and the coefficient functions commute. So D_T satisfies all the symplectic relations.

□

6.8.5 Theorem.

$$D_{\lambda,\mathbf{k}}^{sp} = I_{\lambda,\mathbf{k}}^{sp}.$$

Proof. The dimensions of $D^{sp}_{\lambda,\mathbf{k}}$ and $I^{sp}_{\lambda,\mathbf{k}}$ are both given by Weyl's dimension formula for the symplectic group with respect to λ. So it will be enough to show $D^{sp}_{\lambda,\mathbf{k}} \subset I^{sp}_{\lambda,\mathbf{k}}$.

Let $f \in D^{sp}_{\lambda,\mathbf{k}}$. By Lemma 6.8.4 f satisfies the symplectic relations. Lemma 3.6.4 shows that

$$(f \circ u)(g) = f(g)$$

for all $u \in U^-_{sp}$ and all $g \in Sp_{2n}(\mathbf{k})$. Let $T \subset Sp_{2n}(\mathbf{k})$ be the subgroup of diagonal matrices.

Let $t \in T$. Then t is of the form

$$t = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & 0 & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix}$$

where $t_1, \ldots, t_n \in \mathbf{k} - \{0\}$. Let $g \in Sp_{2n}(\mathbf{k})$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$. Write $g_{i,j}$ for the $(i,j)^{\text{th}}$ coefficient of g. Then

$$(tg)_{i,j} = t_i g_{i,j}.$$

Hence $d_{i,j}(tg) = t_i d_{i,j}(g)$ and $d_{i,j} \circ t = t_i d_{i,j}$.

Let Y be a λ-tableau. Denote by y_i the entry in the i^{th} position of Y. Let $C(\lambda) \subset S_r$ be the permutations which leave stable the columns of Y, and recall that

$$D_Y = \sum_{\sigma \in C(\lambda)} \operatorname{sign}(\sigma) d_{1,y_{\sigma(1)}} \ldots d_{\mu_1, y_{\sigma(\mu_1)}} d_{1, y_{\sigma(\mu_1+1)}} \ldots d_{\mu_2, y_{\sigma(\mu_1+\mu_2)}} \ldots d_{\mu_{\lambda_1}, y_{\sigma(r)}}.$$

We see that D_Y is a sum of monomials $d_{\alpha,\beta}$ where $\alpha \in I(n,r)$, $\beta \in I(\overline{n},r)$ and α contains λ_1 entries equal to 1, λ_2 entries equal to 2, etc. Hence

$$D_Y(tg) = t_1^{\lambda_1} t_2^{\lambda_2} \ldots t_n^{\lambda_n} D_Y(g)$$

for all $g \in Sp_{2n}(\mathbf{k})$. Thus $D_Y(tg) = \lambda(t) D_Y(g)$ for all $g \in Sp_{2n}(\mathbf{k})$ and all $t \in T$.

Let $f \in D^{sp}_{\lambda,\mathbf{k}}$. Then f can be written as a linear combination of elements D_T where T is a λ-tableau. Therefore

$$(f \circ t)(g) = \lambda(t) f(g)$$

for any $t \in \mathbf{k}$. let $b \in B^-$. There is a unique expression for b given by $b = ut$ where $u \in U^-_{sp}$ and $t \in T$. By definition $\lambda(b) = \lambda(t)$. We have shown

$$\begin{aligned}(f \circ b)(g) &= f(bg) \\ &= f(utg) \\ &= f(tg) \\ &= \lambda(t) f(g) \\ &= \lambda(b) f(g)\end{aligned}$$

for all $g \in Sp_{2n}(\mathbf{k})$. Therefore $f \in I^{sp}_{\lambda,\mathbf{k}}$. Hence $D^{sp}_{\lambda,\mathbf{k}} \subset I^{sp}_{\lambda,\mathbf{k}}$, and since they have the same dimension, they must be equal.

□

7

Invariant Forms and Duality.

Let G be a semisimple simply-connected Chevalley group over $\mathbf{k}$. Then G is generated by elements of the form $x_\alpha(t)$, where $t \in \mathbf{k}$ and $\alpha \in \Phi$. In [1] Wong defines an antiautomorphism $\theta : G \to G$ given by

$$\theta(x_\alpha(t)) = x_{-\alpha}(t)$$

for all $t \in \mathbf{k}$ and $\alpha \in \Phi$. When $G = GL_n(\mathbf{k})$, a similar map can be defined, that is, $\theta(g) = g^{tr}$ for all $g \in G$.

Let $n_0 \in Sp_{2n}(\mathbf{k})$ be given by

$$n_0 = \begin{pmatrix} 0 & & & 1 \\ & & \iddots & \\ & & 1 & \\ & 1 & & \\ \iddots & & & \\ 1 & & & 0 \end{pmatrix}$$

where $Sp_{2n}(\mathbf{k})$ is considered as a group of matrices acting on a space with symplectic basis $\{v_1, \ldots, v_n, v_{\overline{n}}, \ldots, v_{\overline{1}}\}$. Then $n_0 \in N(T)$, the normalizer of T in Sp. let π be the canonical map $\pi : N(T) \to N(T)/T = W$ from $N(T)$ to the Weyl group. Then $\pi(n_0) = w_0$ where $w_0 \in W$ is the unique element of maximal length. Then $\theta : Sp_{2n}(\mathbf{k}) \to Sp_{2n}(\mathbf{k})$ given by $\theta(g) = n_0 g^{-1} n_0^{-1}$ for all $g \in Sp_{2n}(\mathbf{k})$ gives an antiautomorphism of $Sp_{2n}(\mathbf{k})$ of the form mentioned above.

7.1 Dual Modules and Bilinear Forms.

We define two different types of dual module. Let V be a rational G-module over $\mathbf{k}$. Let $V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ be the dual to V. Firstly, there is a natural G-action on V^* given by

$$g.f(v) = f(g^{-1}.v)$$

for all $g \in G$, $f \in V^*$ and $v \in V$. This action gives V^* the structure of a rational G-module.

Alternatively, we can define the following G-action on V^*

$$g.f(v) = f(\theta(g).v)$$

for all $g \in G$, $f \in V^*$ and $v \in V$. This also gives a rational G-module structure for V^*. We denote V^* under this action by V^0, and call it the contravariant dual to V.

Let V, W be two G-modules. A bilinear form $(\, , \,) : V \times W \to \mathbf{k}$ is invariant if

$$(g.v, g.w) = (v, w)$$

for all $v \in V$, $w \in W$ and $g \in G$. This is equivalent to

$$(g.v, w) = (v, g^{-1}.w).$$

A bilinear form $< \, , \, >: V \times W \to \mathbf{k}$ is contravariant if

$$(g.v, w) = (v, \theta(g).w)$$

for all $v \in V$, $w \in W$ and $g \in G$.

An invariant form $(\, , \,) : V \times W \to \mathbf{k}$ defines a homomorphism of G-modules $\alpha : V \to W^*$ by

$$\alpha v(w) = (v, w)$$

for all $v \in V$ and $w \in W$. This is an isomorphism if and only if $(\, , \,)$ is non-degenerate. A contravariant form defines a homomorphism of G-modules $\beta : V \to W^0$ by

$$\beta v(w) =< v, w >$$

for all $v \in V$ and $w \in W$, which is an isomorphism if and only if $< \, , \, >$ is non-degenerate.

7.2 Forms in $Sp_{2n}(\mathbf{k})$.

Let $G = Sp_{2n}(\mathbf{k})$ (or Sp for short) and let V be an Sp-module. Let $n_0 \in N$ be as before.

7.2.1 Proposition. *There is a bijective correspondence between the set of invariant forms $(\, , \,) : V \times V \to \mathbf{k}$ and the set of contravariant forms $< \, , \, >: V \times V \to \mathbf{k}$, where $(\, , \,)$ corresponds to $< \, , \, >$ if*

$$(v_1, v_2) =< v_1, n_0.v_2 >$$

for all $v_1, v_2 \in V$.

Proof. Let $(\, , \,)$ and $< \, , \, >$ be any two bilinear forms related by

$$(v_1, v_2) =< v_1, n_0.v_2 >$$

for all $v_1, v_2 \in V$.

Let $g \in G$, $v_1, v_2 \in V$. If $(\ ,\ )$ is invariant then

$$\begin{aligned} < g.v_1, v_2 > &= (g.v_1, n_0^{-1}.v_2) \\ &= (v_1, g^{-1}n_0^{-1}.v_2) \\ &=< v_1, n_0 g^{-1} n_0^{-1}.v_2 > \\ &=< v_1, \theta(g).v_2 > \end{aligned}$$

and $<\ ,\ >$ is contravariant.

Conversely suppose that $<\ ,\ >$ is contravariant. Then

$$\begin{aligned} (g.v_1, g.v_2) &=< g.v_1, n_0 g.v_2 > \\ &=< v_1, \theta(g) n_0 g.v_2 > \\ &=< v_1, n_0 g^{-1} n_0^{-1} n_0 g.v_2 > \\ &=< v_1, n_0.v_2 > \\ &= (v_1, v_2) \end{aligned}$$

and $(\ ,\ )$ is invariant.

□

Hence, when $G = Sp$ a contravariant dual is a dual in the usual sense.

We will show that each of $V^{sp}_{\lambda,\mathbf{k}}$ and $D^{sp}_{\lambda,\mathbf{k}}$ is isomorphic to the dual of the other by constructing a non-degenerate invariant bilinear form $<<\ ,\ >>$: $V^{sp}_{\lambda,\mathbf{k}} \times D^{sp}_{\lambda,\mathbf{k}} \to \mathbf{k}$.

As before, let V be a vector space over $\mathbf{k}$ of dimension $2n$ with a symplectic basis $\{v_1, \ldots, v_n, v_{\overline{n}}, \ldots, v_{\overline{1}}\}$. Define $<\ ,\ >$: $V \times V \to \mathbf{k}$ to be the bilinear form given by the matrix

$$A = \begin{pmatrix} 0 & & & 1 \\ & & \iddots & \\ & & 1 & \\ & -1 & & \\ \iddots & & & 0 \\ -1 & & & \end{pmatrix}$$

So $< v_i, v_j >= A_{i,j}$ for all $i, j \in I \cup \overline{I}$.

This is called the *symplectic form*, and is invariant under the action of Sp on V (by definition). We can extend $<\ ,\ >$ to $T^r(V)$ by defining

$$< v_{i_1} \otimes \ldots \otimes v_{i_r}, v_{j_1} \otimes \ldots \otimes v_{j_r} >=< v_{i_1}, v_{j_1} > \ldots < v_{i_r}, v_{j_r} >$$

for all $i_1, \ldots, i_r, j_1, \ldots, j_r \in I \cup \overline{I}$. This is still invariant under the action of Sp, which acts on each component of a tensor in $T^r(V)$ separately.

Let $\mathcal{T}$ denote the set of all λ-tableaux with entries from $I \cup \overline{I}$, as before. Let $T \in \mathcal{T}$ and let $t_T \in T^r(V)$ be the tensor $v_{t_1} \otimes \ldots \otimes v_{t_r}$ where t_i is the entry in the i^{th} position of T for all $i \in \{1, \ldots, r\}$.

Recall from Chapter 3 the element $\alpha = \sum_{\sigma \in C(\lambda)} \text{sign}(\sigma)\sigma$ in the group algebra $\mathbf{Z}S(r)$ which we use to define a subspace $T^r(V)_\lambda = \alpha T^r(V) \subset T^r(V)$. Recall that $\phi_T \in T^r(V)$ is given by $\phi_T = \alpha t_T$, and $T^R(V)_\lambda$ is the subspace spanned by the tensors ϕ_T for all $T \in \mathcal{T}$.

7.2.2 Definition of the λ-form.
The λ-form $< \, , \, >_\lambda$: $T^r(V) \times T^r(V) \to \mathbf{k}$ is given by

$$< \phi_{T_1}, \phi_{T_2} >_\lambda = < t_{T_1}, \phi_{T_2} >$$

for any λ-tableaux T_1, T_2.

7.2.3 Lemma. *The λ-form is well-defined, and is an invariant bilinear form.*

Proof. Let T_1 and T_2 be λ-tableaux which satisfy $\phi_{T_1} = \phi_{T_2}$. We can assume that in each column of T_1 and T_2 all entries are distinct, otherwise $\phi_{T_1} = \phi_{T_2} = 0$. So $\phi_{T_1} = \alpha t_{T_1}$ is a sum of distinct linearly independent tensors in $T^r(V)$. We have

$$\alpha t_1 = \alpha t_2$$

where $\alpha = \sum_{\sigma \in C(\lambda)} \text{sign}(\sigma)\sigma$ and therefore $t_{T_1} = \text{sign}(\sigma) t_{\sigma(T_2)}$ for some $\sigma \in C(\lambda)$.
Let T be any λ-tableau. Then

$$\begin{aligned} < t_{T_1}, \phi_T > &= \text{sign}(\sigma) < t_{\sigma(T_2)}, \phi_T > \\ &= \sum_{\sigma' \in C(\lambda)} \text{sign}(\sigma')\text{sign}(\sigma) < t_{\sigma(T_2)}, t_{\sigma'(T)} > . \end{aligned}$$

Since $\mathbf{k}$ is commutative under multiplication then this equals

$$\begin{aligned} &\sum_{\sigma' \in C(\lambda)} \text{sign}(\sigma')\text{sign}(\sigma) < t_{\sigma^{-1}\sigma(T_2)}, t_{\sigma^{-1}\sigma'(T)} > \\ &= \sum_{\sigma' \in C(\lambda)} \text{sign}(\sigma^{-1}\sigma') < t_{T_2}, t_{\sigma^{-1}\sigma'(T)} > \\ &= < t_{T_2}, \phi_T > . \end{aligned}$$

So the λ-form is well-defined.
The actions of Sp and of the symmetric group $S(r)$ on $T^r(V)$ commute. For any $t_1, t_2 \in T^r(V)$ and any $g \in Sp$ we have

$$\begin{aligned} < g\alpha t_1, g\alpha t_2 >_\lambda &= < \alpha g t_1, \alpha g t_2 >_\lambda \\ &= < g t_1, \alpha g t_2 > \\ &= < t_1, \alpha t_2 > \\ &= < \alpha t_1, \alpha t_2 >_\lambda \end{aligned}$$

which shows that the λ-form is invariant under the action of Sp.

□

7.2.4 Definitions.

Let the bar map $- : I \cup \bar{I} \to I \cup \bar{I}$ be given by

$$i \mapsto \bar{i}$$
$$\bar{i} \mapsto \bar{\bar{i}} = i$$

for all $i \in I$. In other words, if an element is barred then unbar it, if not then bar it.

We can extend the bar map to $- : \mathcal{T} \to \mathcal{T}$ by putting $\overline{T}$ equal to the unique λ-tableau obtained from T by sending each entry to its image under bar. This is a bijective self-inverse map.

Suppose, for example,

$$T = \begin{array}{|c|c|c|}\hline 1 & \bar{1} & 2 \\ \hline 2 & \bar{2} & \\ \cline{1-2}\end{array}.$$

Then

$$\overline{T} = \begin{array}{|c|c|c|}\hline \bar{1} & 1 & \bar{2} \\ \hline \bar{2} & 2 & \\ \cline{1-2}\end{array}.$$

Let $\mathcal{T}_{col}$ be the set of λ-tableaux with entries strictly increasing down the columns according to the usual ordering

$$1 < \bar{1} < 2 < \bar{2} < \ldots < n < \bar{n}.$$

Let $\mathcal{T}_{\underline{col}}$ be the set of λ-tableaux with entries strictly increasing down the columns according to the unusual ordering

$$\bar{1} < 1 < \bar{2} < 2 < \ldots < \bar{n} < n.$$

Then $- : \mathcal{T}_{col} \to \mathcal{T}_{\underline{col}}$, the restriction of $-$ to $\mathcal{T}_{col}$, is bijective.

7.2.5 Lemma. *Let $T_1, T_2 \in \mathcal{T}$ with $\phi_{T_1} \neq 0$ and $\phi_{T_2} \neq 0$. Then $< \phi_{T_1}, \phi_{T_2} >_\lambda \neq 0$ if and only if $T_1 = \sigma\overline{T_2}$ for some $\sigma \in C(\lambda)$.*

Moreover, if $T_1 = \sigma\overline{T_2}$ for some $\sigma \in C(\lambda)$, then

$$< \phi_{T_1}, \phi_{T_2} >_\lambda = (-1)^{U(T_2)} \mathrm{sign}(\sigma),$$

where $U(T_2)$ is the number of unbarred entries in T_2.

Proof.

$$\begin{aligned} < \phi_{T_1}, \phi_{T_2} >_\lambda &= < t_{T_1}, \phi_{T_2} > \\ &= \sum_{\sigma \in C(\lambda)} \mathrm{sign}(\sigma) < t_{T_1}, t_{\sigma(T_2)} > \end{aligned}$$

Since $\phi_{T_2} \neq 0$, the entries within each column of T_2 are distinct, and hence, as σ runs over $C(\lambda)$ the $\sigma(T_2)$ are all distinct. For a particular σ, $< t_{T_1}, t_{\sigma(T_2)} > \neq 0$ if and only if $T_1 = \overline{\sigma(T_2)} = \sigma(\overline{T_2})$. If $T_1 = \sigma(\overline{T_2})$ for some $\sigma \in C(\lambda)$ and we denote the entries of T_1 by $t_1^1, \ldots, t_r^1$ and those of T_2 by $t_1^2, \ldots, t_r^2$, then

$$\begin{aligned} < \phi_{T_1}, \phi_{T_2} >_\lambda &= \operatorname{sign}(\sigma) < t_{T_1}, t_{\sigma(T_2)} > \\ &= \operatorname{sign}(\sigma) < v_{t_1^1}, v_{t_1^2} > \ldots < v_{t_r^1}, v_{t_r^2} > \\ &= \operatorname{sign}(\sigma)(-1)^{U(T_2)}, \end{aligned}$$

where $U(T_2)$ is the number of i such that t_i^2 is unbarred.

□

7.2.6 Corollary. *Let $T_1 \in \mathcal{T}_{col}$ and $T_2 \in \underline{\mathcal{T}_{col}}$. Then $< \phi_{T_1}, \phi_{T_2} >_\lambda \neq 0$ if and only if $T_1 = \overline{T_2}$, and $< \phi_{T_1}, \phi_{\overline{T_1}} >_\lambda = (-1)^N$ where N is the number of barred entries in T_1.*

7.2.7 Lemma. *The λ-form is non-degenerate.*

Proof. Let $B_1 = \{\phi_T; T \in \mathcal{T}_{col}\}$ and $B_2 = \{\phi_T; T \in \underline{\mathcal{T}_{col}}\}$. Then B_1 and B_2 both form bases for $T^r(V)_\lambda$. We again extend the bar map to give a bijective map $- : B_1 \to B_2$ given by

$$\overline{\phi_T} = \phi_{\overline{T}}.$$

Let M be the matrix of the λ-form with the rows labelled by the elements in B_1 in some order $\phi_{T_1}, \ldots, \phi_{T_s}$ and the columns labelled by the elements of B_2, in the corresponding order $\phi_{\overline{T_1}}, \ldots, \phi_{\overline{T_s}}$. The previous corollary shows M is a diagonal matrix with all entries on the diagonal equal to ± 1. So the the λ-form is non-degenerate.

□

7.2.8 Definition. Let $\theta : T^r(V)_\lambda \to D^{gl}_{\lambda,\mathbf{k}}$ be given by

$$\theta(\phi_T) = b_T$$

for all $T \in \mathcal{T}$, extended by linearity.

7.2.9 Lemma. *The map θ is a well-defined homomorphism of $GL_{2n}(\mathbf{k})$-modules.*

Proof. All of the relations on $T^r(V)_\lambda$ are generated by equations of the form

$$\phi_T = \operatorname{sign}(\sigma)\phi_{\sigma(T)}$$

for some $\sigma \in C(\lambda)$. These equations are also satisfied by the $b_T, T \in \mathcal{T}$, and so θ is well-defined.

Let $i,j \in I \cup \overline{I}$, $i \neq j$ and let $u \in \mathbf{k} - \{0\}$. Denote by $e_{i,j}(u)$ the identity matrix with an extra element u in the $(i,j)^{\text{th}}$ place. Denote by $h_i(u)$ the identity matrix with the element in the $(i,i)^{\text{th}}$ place replaced by $u \in \mathbf{k}$. Such elements generate $GL_{2n}(\mathbf{k})$, and in order to show that θ is a $GL_{2n}(\mathbf{k})$-homomorphism it will be enough to show the actions of these commute with θ.

Let $i \in I \cup \overline{I}$. Then

$$h_i(u)v_j = \begin{cases} v_j & \text{if } j \neq i \\ uv_j & \text{if } j = i \end{cases}.$$

Let $T \in \mathcal{T}$. Then $h_i(u)t_T = u^{N_i(T)}t_T$, where $N_i(T)$ is the number of entries i in T. For any permutation T' of T $N_i(T') = N_i(T)$. So $h_i(u)\phi_T = u^{N_i(T)}\phi_T$ and $\theta(h_i(u)\phi_T) = u^{N_i(T)}b_T$.

Now $\theta(\phi_T) = b_T$ and the action of $h_i(u)$ on b_T is given by

$$h_i(u)b_T(g) = b_T(gh_i(u))$$

for all $g \in GL_{2n}(\mathbf{k})$. Now

$$(gh_i(u))_{k,l} = \begin{cases} g_{k,l} & \text{if } l \neq i \\ ug_{k,l} & \text{if } l = i \end{cases}$$

and so

$$h_i(u)c_{k,l} = \begin{cases} c_{k,l} & \text{if } l \neq i \\ uc_{k,l} & \text{if } l = i \end{cases}.$$

For all $i \in \{1,\ldots,r\}$ let t_i denote the entry in the i^{th} position of T. Then

$$\begin{aligned} h_i(u)b_T &= h_i(u) \sum_{\sigma \in C(\lambda)} \operatorname{sign}(\sigma) c_{1,t_{\sigma(1)}} \cdots c_{r,t_{\sigma(r)}} \\ &= u^N \sum_{\sigma \in C(\lambda)} \operatorname{sign}(\sigma) c_{1,t_{\sigma(1)}} \cdots c_{r,t_{\sigma(r)}}, \end{aligned}$$

where N is the number of ρ such that $t_\rho = i$, i.e. $N = N_i(T)$. So $h_i(u)b_T = u^{N_i(T)}b_T$ and θ commutes with the action of $h_i(u)$.

Let $i,j \in I \cup \overline{I}$. Then

$$e_{i,j}(u)v_k = \begin{cases} v_k & \text{if } k \neq j \\ v_k + uv_i & \text{if } k = j \end{cases}.$$

Let $\alpha = (\alpha_1,\ldots,\alpha_r) \in I(\overline{n},r)$, and write $v_\alpha = v_{\alpha_1} \otimes \ldots \otimes v_{\alpha_r}$. Then

$$e_{i,j}v_\alpha = \sum_{s=0}^{N} u^s \sum_{\gamma} v_{\gamma_1} \otimes \ldots \otimes v_{\gamma_r},$$

where the second sum is over all $\gamma \in I(\overline{n}, r)$ obtained from α by replacing s entries j by i, and where N is the number of ρ such that $\alpha_\rho = j$.

So

$$e_{i,j}(u)t_T = \sum_{s=0}^{N_j(T)} u^s \sum_{T'_s} t_{T'_s},$$

where the second sum is over all T'_s obtained from T by replacing s entries j by i. Hence

$$e_{i,j}(u)\phi_T = \sum_{s=0}^{N_j(T)} u^s \sum_{T'_s} \phi_{T'_s}.$$

So

$$\theta(e_{i,j}(u)\phi_T) = \sum_{s=0}^{N_j(T)} u^s \sum_{T'_s} b_{T'_s},$$

where the second sum is over all T'_s obtained from T by replacing s entries j by i.

Now $\theta(\phi_T) = b_T$, and for all $g \in G$

$$e_{i,j}(u)b_T(g) = b_T(ge_{i,j}(u)),$$

and

$$(ge_{i,j}(u))_{k,l} = \begin{cases} g_{k,l} & \text{if } l \neq j \\ g_{k,l} + ug_{k,i} & \text{if } l = j \end{cases}$$

and so

$$e_{i,j}(u)c_{k,l} = \begin{cases} c_{k,l} & \text{if } l \neq j \\ c_{k,l} + uc_{k,i} & \text{if } l = j \end{cases}.$$

Let $l = (1, 2, \ldots, r)$ and let $\alpha = (\alpha_1, \ldots, \alpha_r) \in I(\overline{n}, r)$. Then $c_{l,\alpha} = c_{1,\alpha_1} \cdots c_{r,\alpha_r}$, and

$$e_{i,j}(u)c_{l,\alpha} = \sum_{s=0}^{N} u^s \sum_{\gamma} c_{l,\gamma}$$

where the second sum is over all $\gamma \in I(\overline{u}, r)$ obtained from α by replacing s entries j by i, and where N is the number of ρ such that $\alpha_\rho = j$.

Hence

$$\begin{aligned} e_{i,j}(u)b_T &= e_{i,j}(u) \sum_{\sigma \in C(\lambda)} c_{1,t_{\sigma(1)}} \cdots c_{r,t_{\sigma(r)}} \\ &= \sum_{s=0}^{N_j(T)} u^s \sum_{T'_s} b_{T'_s}, \end{aligned}$$

where the second sum is over all T'_s obtained from T by replacing s entries j by i. Hence $e_{i,j}(u)\theta(\phi_T) = \theta(e_{i,j}(u)\phi_t)$.

□

7.2.10 Lemma. *$\theta : T^r(V)_\lambda \to D^{gl}_{\lambda,\mathbf{k}}$ is surjective.*

Proof. By definition $D^{gl}_{\lambda,\mathbf{k}}$ is the **k**-span of the elements b_T.

□

7.2.11 Definition. Recall the restriction map $R : D^{gl}_{\lambda,\mathbf{k}} \to D^{sp}_{\lambda,\mathbf{k}} = D^{gl}_{\lambda,\mathbf{k}}|_{sp}$. We define the map $\theta^+ : T^r(V)_\lambda \to D^{sp}_{\lambda,\mathbf{k}}$ to be the composition of $\theta : T^r(V)_\lambda \to D^{gl}_{\lambda,\mathbf{k}}$ and R.

So far we have proved the following.

7.2.12 Lemma. *θ^+ is a surjective $Sp_{2n}(\mathbf{k})$-homomorphism satisfying $\theta^+(\phi_T) = D_T$.*

7.3 The Map δ.

In defining an invariant form on $V^{sp}_\lambda \times D^{sp}_\lambda$, and showing it is non-degenerate, we use a map δ which is a bijection from the subset of symplectic λ-tableaux in $\mathcal{T}$ to itself. The map δ is defined in three stages using various subsets of $\mathcal{T}$ which we shall now describe.

7.3.1 Definitions.

$\mathcal{T}_{ss} = \{T \in \mathcal{T}; T$ is semistandard w.r.t. the usual ordering $1 < \bar{1} < 2 < \bar{2} < \ldots < \bar{n}\}$
$\mathcal{T}_{sp} = \{T \in \mathcal{T}; T$ is symplectic w.r.t. the usual ordering $1 < \bar{1} < 2 < \bar{2} < \ldots < \bar{n}\}$
$\mathcal{T}_{\underline{sp}} = \{T \in \mathcal{T}; T$ is symplectic w.r.t. the unusual ordering $\bar{1} < 1 < \bar{2} < 2 < \ldots < n\}$

Note that the restriction of the bar map $- : \mathcal{T}_{sp} \to \mathcal{T}_{\underline{sp}}$ is a bijection.
We define $U \subset \mathcal{T}$ to be the subset of λ-tableaux T satisfying the following:-

(i) in row r all entries have modulus at least r for all $r \in \{1, \ldots, n\}$;
(ii) the entries of T are strictly increasing down columns according to the usual ordering

$$1 < \bar{1} < \ldots < n < \bar{n};$$

(iii) the rows are non-decreasing with respect to modulus;
(iv) for any $i \in I$ the entries of modulus i in any particular row are ordered as follows. Those which are directly above an entry of modulus i are equal to i, and these will be the left most. Those which are directly below an entry of modulus i are equal to $\bar{i}$, and these will be the right-most. All others will be in the order $\bar{i} < i$ from left to right, and comprise the middle entries.

So the entries of modulus i in a particular row look like the ones in the middle row of this diagram, where $*$ stands for any entry other than i or $\bar{i}$, or possibly a space with no box.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline * & \dots & . & . & \dots & . & . & \dots & * & i & \dots & i \\ \hline i & \dots & i & \bar{i} & \dots & \bar{i} & i & \dots & i & \bar{i} & \dots & \bar{i} \\ \hline \bar{i} & \dots & \bar{i} & * & \dots & . & . & \dots & . & . & \dots & * \\ \hline \end{array}$$

Examples. An example of a tableau in U when $\lambda = (8,7,5)$ and $n = 4$ is

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \bar{1} & 2 & \bar{2} & \bar{2} & 2 & 3 & \bar{3} & \bar{3} \\ \hline \bar{2} & \bar{2} & 3 & \bar{3} & 3 & \bar{3} & \bar{4} \\ \cline{1-7} \bar{3} & \bar{3} & \bar{3} & \bar{4} & 4 \\ \cline{1-5} \end{array}$$

Also if we let $T_1 \in \mathcal{T}_{sp}$ and $T_2 \in \underline{\mathcal{T}_{sp}}$ be

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \bar{2} \\ \hline \end{array} \text{ and } T_2 = \begin{array}{|c|c|}\hline \bar{1} & \bar{2} \\ \hline \bar{2} & 2 \\ \hline \end{array}$$

then $T_2 = \overline{T_1}$.

We can now define some maps which, together with the bar map, compose to form δ.

7.3.2 Definitions.

Let $\kappa : \underline{\mathcal{T}_{sp}} \to U$ be the map which permutes the columns of $T \in \underline{\mathcal{T}_{sp}}$ to be in increasing order according to the usual ordering $1 < \bar{1} < 2 < \bar{2} < \ldots < \bar{n}$.

In particular, the action of κ is to replace any occurrence of $\begin{array}{|c|}\hline \bar{i} \\ \hline i \\ \hline \end{array}$ in a column by $\begin{array}{|c|}\hline i \\ \hline \bar{i} \\ \hline \end{array}$.

It follows that the inverse $\kappa^{-1} : U \to \underline{\mathcal{T}_{sp}}$ is the map which replaces any occurrence of $\begin{array}{|c|}\hline i \\ \hline \bar{i} \\ \hline \end{array}$ in a column of a tableau by $\begin{array}{|c|}\hline \bar{i} \\ \hline i \\ \hline \end{array}$. Therefore κ is a bijection.

Let $\rho : U \to \mathcal{T}_{sp}$ be the map which reorders the rows of a λ-tableau to be in non-decreasing order according to the usual ordering $1 < \bar{1} < 2 < \bar{2} < \ldots < \bar{n}$.

More explicitly, the action of ρ is to reorder the middle entries of a given modulus within in a row, mentioned in condition (iv) of the definition of U. This is illustrated by the following diagram.

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline * & * & * & * & * & * & * & i \\ \hline i & \bar{i} & \dots & \bar{i} & i & \dots & i & \bar{i} \\ \hline \bar{i} & * & * & * & * & * & * & * \\ \hline \end{array} \xrightarrow{\rho} \begin{array}{|c|c|c|c|c|c|c|c|}\hline * & . & * & * & * & * & * & i \\ \hline i & i & \dots & i & \bar{i} & \dots & \bar{i} & \bar{i} \\ \hline \bar{i} & * & * & * & * & * & * & * \\ \hline \end{array}$$

The inverse map $\rho^{-1} : \mathcal{T}_{sp} \to U$ will permute the middle entries back into the order $\bar{i} < i$, and it follows that ρ is a bijection.

7.3.3 Definition of δ.

Let $\delta : \mathcal{T}_{sp} \to \mathcal{T}_{sp}$ be the composition of maps given by

$$\delta(T) = \rho\kappa(\overline{T})$$

for all $T \in \mathcal{T}_{sp}$.

The modulus of an entry in a tableau does not change under δ, only entries of the same modulus are permuted. Also $N_i(\delta(T)) = N_{\bar{i}}(T)$ and $N_{\bar{i}}(\delta(T)) = N_i(T)$. Since the rows of a tableau in $T \in \mathcal{T}_{sp}$ are non-decreasing according to the usual ordering, knowing which entries occur in which rows will determine the tableau uniquely. So once we know which interchanges κ has made, $\delta(T)$ is determined.

In fact, the number of interchanges made by the map κ on a tableau $T \in \mathcal{T}_{sp}$ is equal to the total number of pairs $\{i, \bar{i}\}$ which occur within the same column for all $i \in I$ and for all columns of T. This number is preserved by κ, ρ and the bar map. Let $T \in \mathcal{T}_{sp}$. Then the same number of interchanges are made in $\overline{T}$ under the map κ, as are made in $\overline{\delta(T)}$ under κ. So within each row of $\delta^2(T)$ we have the same number of i's and $\bar{i}$'s as in T, for all $i \in I$. Since $\delta^2(T)$ and T are non-decreasing along rows, we have $\delta^2(T) = T$. Hence δ is self-inverse.

7.3.4 Example.

Let $n = 4$ and $\lambda = (8, 7, 5)$. Let

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & \bar{2} & 3 & 3 & 3 \\ \hline 2 & \bar{2} & 3 & 3 & \bar{3} & \bar{3} & 4 \\ \cline{1-7} 3 & 3 & \bar{3} & 4 & \bar{4} \\ \cline{1-5}\end{array}$$

then

$$\overline{T} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \bar{1} & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{3} & \bar{3} & \bar{3} \\ \hline \bar{2} & 2 & \bar{3} & \bar{3} & 3 & 3 & \bar{4} \\ \cline{1-7} \bar{3} & \bar{3} & 3 & \bar{4} & 4 \\ \cline{1-5}\end{array}$$

$$\kappa(\overline{T}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \bar{1} & 2 & \bar{2} & \bar{2} & 2 & 3 & \bar{3} & \bar{3} \\ \hline \bar{2} & \bar{2} & 3 & \bar{3} & 3 & \bar{3} & \bar{4} \\ \cline{1-7} \bar{3} & \bar{3} & \bar{3} & \bar{4} & 4 \\ \cline{1-5}\end{array}$$

and

$$\delta(T) = \rho\kappa(\overline{T}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \bar{1} & 2 & 2 & \bar{2} & \bar{2} & 3 & \bar{3} & \bar{3} \\ \hline \bar{2} & \bar{2} & 3 & 3 & \bar{3} & \bar{3} & \bar{4} \\ \cline{1-7} \bar{3} & \bar{3} & \bar{3} & 4 & \bar{4} \\ \cline{1-5}\end{array}$$

$$\kappa(\overline{\delta(T)}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & \bar{2} & 2 & 2 & 3 & 3 & 3 \\ \hline 2 & \bar{2} & 3 & \bar{3} & 3 & \bar{3} & 4 \\ \cline{1-7} 3 & 3 & \bar{3} & \bar{4} & 4 \\ \cline{1-5}\end{array}$$

and finally

$$\delta^2(T) = \rho\kappa(\overline{\delta(T)}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & \bar{2} & 3 & 3 & 3 \\ \hline 2 & \bar{2} & 3 & 3 & \bar{3} & \bar{3} & 4 \\ \cline{1-7} 3 & 3 & \bar{3} & 4 & \bar{4} \\ \cline{1-5}\end{array} = T.$$

Note that the number of interchanges made by κ in T is equal to the total number of column repeats in T. Let R_i denote the repeat set of column i of T, and let

$$\mathcal{R}(T) = |R_1| + \ldots + |R_{\lambda_1}|.$$

Then $\mathcal{R}(T)$ is the number of interchanges there will be in $\overline{T}$ under the map κ. For any $T \in \mathcal{T}$ let $U(T)$ (respectively $B(T)$) denote the total number of unbarred (respectively barred) entries in T. These are related by the equation $U(T) + B(T) = r$.

7.3.5 Lemma. *Let $T \in \mathcal{T}_{sp}$. Then*

$$< \psi_{\delta(T)}, \phi_T >_\lambda = (-1)^{U(T)+\mathcal{R}(T)}.$$

Proof.

$$\begin{aligned} < \psi_{\delta(T)}, \phi_T >_\lambda &= \sum_{T'} < \phi_{T'}, \phi_T >_\lambda \\ &= \sum_{T'} < t_{T'}, \phi_T > \end{aligned}$$

where the sum is over all distinct row permutations T' of $\delta(T)$ which satisfy the condition that within any column of T' all the entries are distinct (otherwise $\phi_{T'} = 0$). $T' = \rho^{-1}(\delta(T))$ satisfies these conditions.

We have $< t_{T'}, \phi_T > \neq 0$ if and only if $T' = \sigma(\overline{T})$ for some column permutation $\sigma \in C(\lambda)$. By Lemma 7.2.5 if $T' = \sigma(\overline{T})$ for $\sigma \in C(\lambda)$ then

$$< t_{T'}, \phi_T > = (-1)^{U(T)}\mathrm{sign}(\sigma).$$

When $T' = \rho^{-1}(\delta(T))$ then $T' = \kappa(\overline{T})$ and so

$$< t_{\rho^{-1}(\delta(T))}, \phi_T > = (-1)^{U(T)+\mathcal{R}(T)}.$$

Suppose there was another row permutation T' of $\delta(T)$ such that there exists $\sigma \in C(\lambda)$ with $T' = \sigma(\overline{T})$. Since T' is a row permutation of $\delta(T)$ it must have the same number of i's and of $\bar{i}$'s as $\delta(T)$ in each row for all $i \in I$. So σ must move the same number of i's in and out of each row, and of $\bar{i}$'s in and out of each row, of $\overline{T}$ as κ does.

The only way that i and $\bar{i}$ occur in the same column is immediately above and below each other, and κ interchanges all such occurrences. Hence κ makes the maximum number of such interchanges and is therefore the unique way to do so. So no such T' exists, and

$$< \psi_{\delta(T)}, \phi_T >_\lambda = < t_{\rho^{-1}(\delta(T))}, \phi_T > = (-1)^{U(T)+\mathcal{R}(T)}.$$

□

7.3.6 Corollary. *Let $T \in \mathcal{T}_{sp}$. Then*

$$< \psi_T, \phi_{\delta(T)} >_\lambda = (-1)^{B(T)+\mathcal{R}(T)}.$$

Proof. This follows because $\delta^2(T) = T$ and $U(T) = B(\delta(T))$.

□

7.4 The Invariant Form.

Let T be a non-symplectic semistandard λ-tableau. In Chapter 6 we used the compound relations to obtain the expression

$$D_T = \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} D_{T'},$$

where the $c_{T,T'} \in \mathbf{k}$ and each T' is a symplectic λ-tableau.

7.4.1 Lemma. *Recall the map $R : D^{gl}_{\lambda,\mathbf{k}} \to D^{gl}_{\lambda,\mathbf{k}}|_{Sp} = D^{sp}_{\lambda,\mathbf{k}}$. Let $v \in \ker R$. Then v is a linear combination of elements*

$$b_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} b_{T'},$$

where $T \in \mathcal{T}_{ss}$.

Proof. Let $T \in \mathcal{T}_{ss}$. Then

$$R(b_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} b_{T'}) = D_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} D_{T'} = 0.$$

Hence, $b_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} b_{T'}$ is in the kernel of R.

The set $\{b_T; T \in \mathcal{T}_{ss}\}$ forms a basis for $D^{gl}_{\lambda,\mathbf{k}}$, and any element $v \in \ker R$ can be written

$$v = \sum_{T \in \mathcal{T}_{ss}} \alpha_T b_T$$

for $\alpha \in \mathbf{k}$. Let $T^* \in \mathcal{T}_{ss} \setminus \mathcal{T}_{sp}$ be a semistandard non-symplectic λ-tableau which satisfies $\alpha_{T^*} \neq 0$. We can construct a new element $v' \in \mathit{ker} R$ by letting

$$v' = v - \alpha_{T^*}(b_{T^*} - \sum_{T' \in \mathcal{T}_{sp}} c_{T^*,T'} b_{T'}).$$

The number of semistandard non-symplectic λ-tableaux which have non-zero coefficients is one less in v' than in v. We can continue repeating this process until we obtain $v'' \in \ker R$ such that

$$v'' = \sum_{T'' \in \mathcal{T}_{sp}} \alpha''_{T''} b_{T''}$$

where $\alpha''_{T''} \in \mathbf{k}$, satisfying $v = v''+$ a linear combination of elements of the form

$$b_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} b_{T'}.$$

So $v'' \in Ker(R)$. However, $R(v'') = \sum_{T'' \in \mathcal{T}_{sp}} \alpha''_{T''} D_{T''}$, but the set $\{D_T \ ; \ T \in \mathcal{T}_{sp}\}$ is linearly independent. Therefore, all the coefficients $\alpha''{}_{T''}$ are zero, and $v'' = 0$.

□

7.4.2 Definitions.
Let μ be the conjugate partition to λ, and for $j \in \{1, \ldots, \lambda_1\}$ let

$$I_j = \{i \in \mathbb{N}; \mu_1 + \ldots + \mu_{j-1} < i \leq \mu_1 + \ldots + \mu_j\}.$$

So I_j can be regarded as the set of squares in the j^{th} column of the λ-diagram. Suppose we have an abelian group A with an element $x_T \in A$ corresponding to T for all $T \in \mathcal{T}$. Let $T \in \mathcal{T}$ and for $h \in \{1, \ldots \lambda_1 - 1\}$ let $J_h \subset I_h$ and $J_{h+1} \subset I_{h+1}$ satisfy $|J_h| + |J_{h+1}| > |I_h|$. Then we define a *Garnir element* in A corresponding to $T \in \mathcal{T}$ and the subsets J_h and J_{h+1} by

$$G(x_T, J_h, J_{h+1}) = \sum_{\sigma \in S(J)} \operatorname{sign}(\sigma) x_{\sigma(T)}$$

where σ runs over the set $S(J)$ of all permutations of $\{1, 2, \ldots, r\}$ which are the identity outside $J_h \cup J_{h+1}$ and such that $\sigma(i) < \sigma(j)$ for $i < j$ in J_h and for $i < j$ in I_{h+1}.

An equation of the form

$$G(x_T, J_h, J_{h+1}) = 0$$

is a *Garnir relation* in A. If the elements $x_T \in A$ satisfy equations of this form for all $T \in \mathcal{T}$ and all suitable subsets I_h and I_{h+1} for all h, then A is said to satisfy the Garnir relations.

The Garnir relations were defined by Carter and Lusztig in [1], in which they show that $V^{gl}_{\lambda,\mathbf{k}}$ satisfies the Garnir relations in terms of the elements $\psi_T \in V^{gl}_{\lambda,\mathbf{k}}$. Green shows in [1] that the Garnir relations are satisfied in $D^{gl}_{\lambda,\mathbf{k}}$ in terms of the elements $b_T \in D^{gl}_{\lambda,\mathbf{k}}$, from which it immediately follows that $D^{sp}_{\lambda,\mathbf{k}}$ satisfies the Garnir relations in terms of the elements $D_T \in D^{sp}_{\lambda,\mathbf{k}}$.

7.4.3 Lemma. *Let $T_1, T_2 \in \mathcal{T}_{ss}$. Let $G(\phi_{T_2}, J_h, J_{h+1})$ be a Garnir element in $T^r(V)_\lambda$. Then*

$$< \psi_{T_1}, G(\phi_{T_2}, J_h, J_{h+1}) >_\lambda = 0.$$

Proof. Let $\mathcal{S} \subset \mathcal{T}$ be the set of all row permutations of T_1 with no repeats, and let S(J) be the set of all permutations σ of $\{1,2,\ldots,r\}$ which are the identity outside $J_h \cup J_{h+1}$ and such that $\sigma(i) < \sigma(j)$ for $i < j$ in J_h and for $i < j$ in J_{h+1}. Then

$$\begin{aligned}
< \psi_{T_1}, G(\phi_{T_2}, J_h, J_{h+1}) >_\lambda &= \sum_{T\in\mathcal{S}} < \phi_T, G(\phi_{T_2}, J_h, J_{h+1}) >_\lambda \\
&= \sum_{T\in\mathcal{S}} < t_T, G(\phi_{T_2}, J_h, J_{h+1}) > \\
&= \sum_{T\in\mathcal{S}} \sum_{\sigma\in S(J)} \mathrm{sign}(\sigma) < t_T, \phi_{\sigma(T_2)} > \\
&= \sum_{T\in\mathcal{S}} \sum_{\sigma\in S(J)} \sum_{\rho\in C(\lambda)} \mathrm{sign}(\sigma)\mathrm{sign}(\rho) < t_T, t_{\rho\sigma(T_2)} > \\
&= \sum_{\rho\in C(\lambda)} \mathrm{sign}(\rho) \left[\sum_{\sigma\in S(J)} \mathrm{sign}(\sigma) \sum_{T\in\mathcal{S}} < t_T, t_{\rho\sigma(T_2)} > \right]
\end{aligned}$$

Since **k** is commutative, for any $T_1^*, T_2^* \in \mathcal{T}$ and any $\gamma \in S(r)$

$$< t_{T_1^*}, t_{T_2^*} > = < t_{\gamma T_1^*}, t_{\gamma T_2^*} > .$$

So the above is equal to

$$\sum_{\rho\in C(\lambda)} \mathrm{sign}(\rho) \left[\sum_{\sigma\in S(J)} \mathrm{sign}(\sigma) \sum_{T\in\mathcal{S}} < t_{\rho^{-1}T}, t_{\sigma(T_2)} > \right]$$

If the entries of T_2 in the squares in $J_h \cup J_{h+1}$ are not all distinct then $G(\phi_{T_2}, J_h, J_{h+1}) = 0$. In this case the lemma follows immediately. So assume that the entries of T_2 in the squares in $J_h \cup J_{h+1}$ are all distinct. Then, by the definition of $< \, , \, >$ for any $\rho \in C(\lambda)$ and any $\sigma \in S(J)$

$$< t_{\rho^{-1}T}, t_{\sigma(T_2)} > = 0$$

unless the entries of $\rho^{-1}(T)$ in squares in $J_h \cup J_{h+1}$ are all distinct. Fix $\rho \in C(\lambda)$ and let $\mathcal{S}' \subset \mathcal{S}$ be the set of $T \in \mathcal{S}$ with all entries distinct in the squares in $J_h \cup J_{h+1}$. Then

$$\sum_{\sigma\in S(J)} \mathrm{sign}(\sigma) \sum_{T\in\mathcal{S}} < t_{\rho^{-1}T}, t_{\sigma(T_2)} > = \sum_{\sigma\in S(J)} \mathrm{sign}(\sigma) \sum_{T\in\mathcal{S}'} < t_{\rho^{-1}T}, t_{\sigma(T_2)} > .$$

Let $\mathcal{S}^* = \{\rho^{-1}(T); T \in \mathcal{S}'\}$. Then the above is equal to

$$\sum_{\sigma\in S(J)} \mathrm{sign}(\sigma) \sum_{T\in\mathcal{S}^*} < t_T, t_{\sigma(T_2)} > . \quad (*)$$

The two column subsets J_h and J_{h+1} must overlap at some row since $|J_h| + |J_{h+1}| > |I_h|$. That is, there must be at least one row containing squares in both J_h and J_{h+1}. Let

$\sigma^* \in S(J)$ be the transposition which interchanges the entries in these two squares. Let CR denote the set of right coset representatives of the subgroup $\{1, \sigma^*\}$ in $S(J)$. Since σ is a transposition it has sign -1, and we can rewrite equation $(*)$ as

$$\sum_{\sigma \in CR} \text{sign}(\sigma) \sum_{T \in \mathcal{S}^*} (< t_T, t_{\sigma(T_2)} > - < t_T, t_{\sigma^* \sigma(T_2)} >).$$

For any $T \in \mathcal{S}^*$ we have $\sigma^*(T) \neq T$ because the entries of T in squares $J_h \cup J_{h+1}$ are all distinct. So we can partition $\mathcal{S}^*$ into m disjoint subsets, for some $m \in \mathbb{N}$,

$$\mathcal{S}^* = \{T^1, \sigma^*(T^1)\} \sqcup \{T^2, \sigma^*(T^2)\} \sqcup \ldots \sqcup \{T^m, \sigma^*(T^m)\}$$

and rewrite equation $(*)$ as

$$\sum_{\sigma \in CR} \text{sign}(\sigma) \sum_{i=1}^{m} \begin{pmatrix} < t_{T^i}, t_{\sigma(T_2)} > - < t_{T^i}, t_{\sigma^* \sigma(T_2)} > + < t_{\sigma^*(T^i)}, t_{\sigma(T_2)} > \\ \qquad - < t_{\sigma^*(T^i)}, t_{\sigma^* \sigma(T_2)} > \end{pmatrix}.$$

However $< t_{T^i}, t_{\sigma(T_2)} > = < t_{\sigma^*(T^i)}, t_{\sigma^* \sigma(T_2)} >$ and $< t_{T^i}, t_{\sigma^* \sigma(T_2)} > = < t_{\sigma^*(T^i)}, t_{\sigma(T_2)} >$ since $\sigma^* = (\sigma^*)^{-1}$.

Hence the terms in the sum cancel in pairs and altogether the sum is equal to zero, proving the lemma.

□

Recall the map $\theta : T^r(V)_\lambda \to D^{gl}_{\lambda, \mathbf{k}}$ which sends ϕ_T to b_T for all $T \in \mathcal{T}$.

7.4.4 Lemma. *Let $v \in \ker \theta \subset T^r(V)_\lambda$ and $v' \in V^{sp}_{\lambda, \mathbf{k}}$. Then*

$$< v', v >_\lambda = 0.$$

Proof. The set $\{\phi_T; T \in \mathcal{T}_{col}\}$ forms a basis for $T^r(V)_\lambda$. In Carter and Lusztig [1] the Garnir relations are used to find the expression

$$b_T = \sum_{T' \in \mathcal{T}_{ss}} \mu_{T,T'} b_{T'},$$

where the $\mu_{T,T'} \in \mathbf{k}$, for any $T \in \mathcal{T}_{col}$. By a similar argument to that in Lemma 7.4.1 we know that v is a linear combination of elements of the form

$$\phi_T - \sum_{\mathcal{T}_{ss}} \mu_{T,T'} \phi_{T'}.$$

These elements are linear combinations of Garnir elements in $T^r(V)_\lambda$, and so it will be enough to show for any Garnir element $G(\phi_T, J_h, J_{h+1})$

$$< v', G(\phi_T, J_h, J_{h+1}) >_\lambda = 0$$

for all $v' \in V^{sp}_{\lambda,\mathbf{k}}$. The set $\{\psi_T : T \in \mathcal{T}_{ss}\}$ forms a basis for $V^{gl}_{\lambda,\mathbf{k}}$, and $V^{sp}_{\lambda,\mathbf{k}} \subset V^{gl}_{\lambda,\mathbf{k}}$, and hence v' can be written as a linear combination of elements ψ_T for $T \in \mathcal{T}_{ss}$. So it will be enough to show that for any $T_1 \in \mathcal{T}_{ss}$ and any Garnir element $G(\phi_{T_2}, J_h, J_{h+1})$

$$< \psi_{T_1}, G(\phi_{T_2}, J_h, J_{h+1}) >_\lambda = 0,$$

which is true by Lemma 7.4.3.

□

Recall the map $\theta^+ : T^r(V)_\lambda \longrightarrow D^{sp}_{\lambda,\mathbf{k}}$ which sends ϕ_T to D_T for all $T \in \mathcal{T}_{ss}$.

7.4.5 Lemma. *Let $v \in \mathrm{Ker}\theta^+$. Then for all $v' \in V^{sp}_{\lambda,\mathbf{k}}$*

$$< v', v >_\lambda = 0.$$

Proof. Since $v \in \mathrm{Ker}\theta^+$, we have $\theta(v) \in \mathrm{Ker}R$. By Lemma 7.4.4 $\theta(v)$ is a linear combination of elements of the form

$$b_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} b_{T'}.$$

Hence $v = v_1 + v_2$ where $v_1 \in \ker\theta$ and v_2 is a linear combination of elements of the form

$$\phi_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} \phi_{T'}$$

with $T \in \mathcal{T}_{ss}$.

So

$$< v', v >_\lambda = < v', v_1 >_\lambda + < v', v_2 >_\lambda = < v', v_2 >_\lambda$$

by Lemma 7.2.9. Hence it is enough to show for all $v' \in V^{sp}_\lambda$ and $T \in \mathcal{T}_{ss}$

$$< v', \ \phi_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} \phi_{T'} >_\lambda = 0.$$

Recall that in chapter 6 we originally obtained the expression $D_T = \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} D_{T'}$ for any $T \in \mathcal{T}_{ss}$ using the compound relations. Hence

$$\phi_T - \sum_{T' \in \mathcal{T}_{sp}} c_{T,T'} \phi_{T'}$$

is a linear combination of elements in the images of various expansion operators. Recall that a $\lambda_{s,t}$-tableau is a diagram which becomes a λ-tableau when $2t$ squares (and their entries) are added to the bottom of the s^{th} column. So it will be enough to prove for any $v' \in V^{sp}_\lambda$ and any $\lambda_{s,t}$-tableau Y, where $s \in \{1, 2, \ldots, \lambda_1 + 1\}$ and $t \in \mathbb{N}$ satisfies $2t \leq \mu_s$, that $< v', E_{s,t}(\phi_Y) >_\lambda = 0$, where $E_{s,t}$ is the expansion operator adding $2t$ squares to the bottom of column s.

Let Y be a $\lambda_{s,t}$-tableau. The expansion operator $E_{s,t}$ maps ϕ_Y to $E_{s,t}(\phi_Y) \in T^r(V)_\lambda$ given by

$$E_{s,t}(\phi_Y) = \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} \phi_{Y^s(i_1,\ldots,i_t)}$$

where $Y^s(i_1,..,i_t)$ denotes the λ-tableau obtained by adding $2t$ squares to the s^{th} column of Y and putting in the entries $i_1, \bar{i}_1, i_2, \bar{i}_2, \ldots, i_t, \bar{i}_t$ in order.

As $v' \in T^r(V)_\lambda$ it can be expressed in the form

$$v' = \sum_{T' \in \mathcal{T}_{col}} k_{T'} \phi_{T'}$$

where, for all $T' \in \mathcal{T}_{col}$, $k_{T'} \in \mathbf{k}$. So

$$< v', E_{s,t}(\phi_Y) >_\lambda = \left\langle \sum_{T' \in \mathcal{T}_{col}} k_{T'} \phi_{T'} \, , \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} \phi_{Y^s(i_1,\ldots,i_t)} \right\rangle_\lambda .$$

By Lemma 7.2.5 we have that for each choice of $i_1, \ldots, i_t \in I$ there is a unique λ-tableau $T^s(i_1, \ldots, i_t) \in \mathcal{T}_{col}$ such that $< \phi_{T^s(i_1,\ldots,i_t)}, \phi_{Y^s(i_1,\ldots,i_t)} >_\lambda \neq 0$, and that is

$$T^s(i_1, \ldots, i_t) = \sigma(\overline{Y^s(i_1, \ldots, i_t)})$$

where $\sigma \in C(\lambda)$ permutes the columns of $\overline{Y^s(i_1, \ldots, i_t)}$ to make them strictly increasing. In this case

$$< \phi_{T^s(i_1,\ldots,i_t)}, \phi_{Y^s(i_1,\ldots,i_t)} >_\lambda = (-1)^{U(T^s(i_1,\ldots,i_t))} \text{sign}(\sigma).$$

The permutation σ has two components, one which interchanges repeats (which were previously interchanged by the bar map) and one which reorders the remaining entries of Y and the new entries $i_1, \bar{i}_1, \ldots, i_t, \bar{i}_t$ to be in strictly increasing order. So sign(σ) is independent of the choice of $i_1, \ldots, i_t \in I$ since permuting adjacent pairs always involves an even permutation. Also $U(T^s(i_1, \ldots, i_t))$ is clearly independent of the choice of $i_1, \ldots, i_t$. So

$$\begin{aligned} < v', E_{s,t}(\phi_Y) >_\lambda &= \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} k_{T^s(i_1,\ldots,i_t)} < \phi_{T^s(i_1,\ldots,i_t)}, \phi_{Y^s(i_1,\ldots,i_t)} >_\lambda \\ &= \pm \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} k_{T^s(i_1,\ldots,i_t)}. \end{aligned}$$

Let $\Omega_{s,t} : T^r(V)_\lambda \to T^{r-2s}(V)$ be the contraction mapping of degree t which acts on the s^{th} column. Then $\sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} k_{T^s(i_1,\ldots,i_t)}$ is the coefficient of ϕ_Y in $\Omega_{s,t}(v')$. Since $v' \in V_\lambda^{sp}$ it is traceless, and so

$$< v', E_{s,t}(\phi_Y) >_\lambda = \pm \sum_{\substack{i_1,\ldots,i_t \in I \\ i_1 < \ldots < i_t}} k_{T^s(i_1,\ldots,i_t)} = 0.$$

□

7.4.6 Definition. We use the fact that $\theta^+ : T^r(V)_\lambda \to D_\lambda^{sp}$ is surjective to define the bilinear form $<< \, , \, >>: V_\lambda^{sp} \times D_\lambda^{sp} \to \mathbf{k}$ by

$$<< v', \theta^+(v) >> = < v', v >_\lambda .$$

7.4.7 Lemma. $<< \, , \, >>: V_\lambda^{sp} \times D_\lambda^{sp} \to \mathbf{k}$ *is a well-defined Sp-invariant bilinear form.*

Proof. It is well-defined by Lemma 7.4.5, and the Sp-invariance follows from the fact that θ^+ is an $Sp_{2n}(\mathbf{k})$-homomorphism, and the λ-form is invariant.

□

7.4.8 Lemma. *For all $T \in \mathcal{T}_{sp}$*

$$<< v_T, D_{\delta(T)} >> = \pm 1.$$

Proof. Let $T \in \mathcal{T}_{sp}$. Recall from Chapter 5 that $v_T \in V_\lambda^{sp}$ is of the form

$$v_T = \psi_T + \sum_{T' \in \mathcal{T}_s} k_{T'} \psi_{T'}$$

where $k_{T'} = 0$ for all T' such that $\mathrm{ht}(T') \geq \mathrm{ht}(T)$. Hence, for all T' such that $k_{T'} \neq 0$ $\mathrm{ht}(T') < \mathrm{ht}(\delta(T))$, T' cannot be a permutation of $\delta(T)$. and

$$<< \psi_{T'}, D_{\delta(T)} >> = < \psi_{T'}, \phi_{\delta(T)} >_\lambda = 0.$$

Therefore,

$$<< v_T, D_{\delta(T)} >> = < v_T, \phi_{\delta(T)} >_\lambda = < \psi_T + \sum_{T'} k_{T'} \psi_{T'}, \phi_{\delta(T)} >_\lambda = < \psi_T, \phi_{\delta(T)} >_\lambda .$$

By Corollary 7.3.6 we have

$$< \psi_T, \phi_{\delta(T)} >_\lambda = \pm 1.$$

□

7.4.9 Definition. We introduce a total ordering on the set of symplectic λ-tableaux by defining, for all $T_1, T_2 \in \mathcal{T}_{sp}$ such that $T_1 \neq T_2$, $T_1 < T_2$ if either of the following two conditions are satisfied:-

(i) $\mathrm{ht}(T_1) < \mathrm{ht}(T_2)$;
(ii) $\mathrm{ht}(T_1) = \mathrm{ht}(T_2)$ and the first row which is not the same in T_1 and T_2 has left-most differing element greater in T_2.

7.4.10 Lemma. *Let $T_1, T_2 \in \mathcal{T}_{sp}$ with $T_1 \neq T_2$. Then*

$$<< \psi_{T_1}, D_{\delta(T_2)} >> \neq 0 \;\Rightarrow\; T_1 > T_2.$$

Proof. Let $T_3 = \delta(T_2)$.

$$\begin{aligned} << \psi_{T_1}, D_{T_3} >> &= < \psi_{T_1}, \phi_{T_3} >_\lambda \\ &= \sum_{T_1'} < \phi_{T_1'}, \phi_{T_3} >_\lambda \end{aligned}$$

where the sum is over the set of row permutations T_1' of T_1.

Since $<< \psi_{T_1}, D_{T_3} >> \neq 0$, at least one T_1' must exist such that

$$< \phi_{T_1'}, \phi_{T_3} >_\lambda \neq 0,$$

and so, by Lemma 7.2.5 there is a $\sigma \in C(\lambda)$ such that $T_1' = \sigma(\overline{T_3})$. Hence T_1 is equal to a row permutation γ of a column permutation σ of $\overline{T_3}$. The map κ sends $\overline{T_3}$ to a permutation of itself, and we shall also call the permutation κ. T_3 and the column permutation σ determine T_1 since the row permutation must leave it non-decreasing along rows. If $\sigma = \kappa$ then

$$T_1 = \gamma\sigma(\overline{T_3}) = \delta(T_3) = \delta^2(T_2) = T_2,$$

which contradicts $T_1 \neq T_2$. So $\sigma \neq \kappa$.

However, κ is the unique column permutation of $\overline{T_3}$ which leaves the columns in strictly increasing order. Hence the columns of $\sigma(\overline{T_3})$ are not in increasing order. Therefore the highest row on which $\kappa(\overline{T_3})$ and $\sigma(\overline{T_3})$ differ does so by having greater elements in $\sigma(\overline{T_3})$.

Since T_1 is a permutation of $\delta(T_3)$ they have the same height, and by the above they satisfy condition (ii) of Definition 7.4.9 for $T_1 > \delta(T_3)$. Hence

$$T_1 > \delta(T_3) = \delta^2(T_2) = T_2.$$

□

7.4.11 Lemma. *Let $T_1, T_2 \in \mathcal{T}_{Sp}$ Then:-*
(i) $<< v_{T_1}, D_{\delta(T_2)} >> \neq 0 \Rightarrow T_1 \geq T_2$;
(ii) $T_1 = T_2 \Rightarrow << v_{T_1}, D_{\delta(T_2)} >> = \pm 1$.

Proof. Let $T_1, T_2 \in \mathcal{T}_{Sp}$. Part (ii) follows from Lemma 7.4.8, so we will assume $T_1 \neq T_2$. Suppose $<< v_{T_1}, D_{\delta(T_2)} >> \neq 0$. Then

$$v_{T_1} = \psi_{T_1} + \sum_{T_1'} k_{T_1'} \psi_{T_1'}$$

such that whenever T_1' satisfies $k_{T_1'} \neq 0$, $\mathrm{ht}(T_1') < \mathrm{ht}(T_1)$, and

$$\begin{aligned} << v_{T_1}, D_{\delta(T_2)} >> &= < \psi_{T_1} + \sum_{T_1'} k_{T_1'} \psi_{T_1'} \ , \ \phi_{\delta(T_2)} >_\lambda \\ &= < \psi_{T_1}, \phi_{\delta(T_2)} >_\lambda + \sum_{T_1'} k_{T_1'} < \psi_{T_1'} \ , \ \phi_{\delta(T_2)} >_\lambda . \end{aligned}$$

If $<< \psi_{T_1}, \phi_{\delta(T_2)} >_\lambda \neq 0$ then $T_1 > T_2$, by Lemma 7.4.10. Otherwise, if $< \psi_{T_1}, \phi_{\delta(T_2)} >_\lambda = 0$ then

$$< \psi_{T_1'}, \phi_{\delta(T_2)} >_\lambda \neq 0$$

for some T_1' such that $\mathrm{ht}(T_1') < \mathrm{ht}(T_1)$. In this case T_1' is a row permutation of a column permutation of $\overline{\delta(T_2)}$ and so $\mathrm{ht}(T_1') = \mathrm{ht}(\overline{\delta(T_2)}) = \mathrm{ht}(T_2)$. Thus $\mathrm{ht}(T_2) < \mathrm{ht}(T_1)$ and $T_1 > T_2$.

□

7.4.12 Definition. Write the set $\mathcal{T}_{sp}$ in the form $\mathcal{T}_{sp} = \{T_1, T_2, \ldots, T_q\}$ where $q = |\mathcal{T}_{sp}|$ and $T_1 < \ldots < T_q$ according to the total ordering in Definition 7.4.9. For $T, T' \in \mathcal{T}_{sp}$ define $m_{T,T'} = << v_T, D_{T'} >>$. Let M be the matrix over $\mathbf{k}$ with rows and columns indexed by $\mathcal{T}_{sp}$ given by

$$(M)_{T,T'} \ = \ m_{T,T'}$$

where the rows are in the order $T_1, T_2, \ldots, T_q$ and the columns are in the corresponding order $\delta(T_1), \delta(T_2), \ldots, \delta(T_q)$. So M is the matrix of the form $<<, >>$.

7.4.13 Lemma. *The form* $<< \ , \ >>: V^{sp}_{\lambda,\mathbf{k}} \times D^{sp}_{\lambda,\mathbf{k}} \to \mathbf{k}$ *is non-degenerate.*

Proof. The form $<<, >>$ is non-degenerate if and only if its matrix M is non-singular. By Lemma 7.4.11 M has all diagonal coefficients equal to ± 1, and has all coefficients above the diagonal equal to zero. Hence $\det M = \pm 1$ and M is non-singular.

□

7.4.14 Theorem. *Each of* $V^{sp}_{\lambda,\mathbf{k}}$ *and* $D^{sp}_{\lambda,\mathbf{k}}$ *is isomorphic to the dual of the other.*

Proof. The form $<<\ ,\ >>: V^{sp}_{\lambda,\mathbf{k}} \times D^{sp}_{\lambda,\mathbf{k}} \to \mathbf{k}$ is invariant and non-degenerate by Lemma 7.4.13, and the result follows.

□

Recall that $V^{sp}_{\lambda,\mathbf{k}}$ has a unique maximal submodule $M^{sp}_{\lambda,\mathbf{k}}$ and the factor module

$$F^{sp}_{\lambda,\mathbf{k}} = V^{sp}_{\lambda,\mathbf{k}}/M^{sp}_{\lambda,\mathbf{k}}$$

is irreducible of highest weight λ. By the above theorem we know that $D^{sp}_{\lambda,\mathbf{k}}$ has a unique minimal submodule isomorphic to $F^{sp}_{\lambda,\mathbf{k}}$.

7.5 The Complete Set of Modules.

7.5.1 Definition. Let $\mathbf{k}$ be any infinite field. The additive group of $\mathbf{k}$ is an affine algebraic group, and can therefore be written in terms of matrices. We do this by defining the group of matrices $\mathbf{k}^+$ by

$$\mathbf{k}^+ = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} : \mu \in \mathbf{k} \right\}.$$

7.5.2 Lemma. *When* $\mathrm{char}(\mathbf{k}) = 0$ *the only irreducible polynomial representation of* $\mathbf{k}^+$ *is the unit representation.*

Proof. Let V be a $\mathbf{k}^+$-module affording a polynomial representation

$$\rho : \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \mapsto \rho(\mu) \in Gl_n(\mathbf{k}) \quad \textit{for some } n \in \mathbb{N}.$$

The coordinate function

$$\rho_{i,j} : \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \mapsto \rho(\mu)_{i,j}$$

is a polynomial in μ, and so

$$\rho_{i,j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \sum_{l=0}^{\infty} \mu^l c^l_{i,j}$$

for some $c^l_{i,j} \in \mathbf{k}$ where all but a finite number are zero. For $l \in \mathbb{N}$ let $C^{(l)}$ be the matrix given by

$$(C^{(l)})_{i,j} = c^l_{i,j}.$$

Then

$$\rho \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \sum_{l=0}^{\infty} \mu^l C^{(l)}.$$

Since ρ is a homomorphism

$$\rho\begin{pmatrix} 1 & \mu_1 \\ 0 & 1 \end{pmatrix} \rho\begin{pmatrix} 1 & \mu_2 \\ 0 & 1 \end{pmatrix} = \rho\begin{pmatrix} 1 & \mu_1 + \mu_2 \\ 0 & 1 \end{pmatrix}$$

and so

$$\left(\sum_{l=0}^{\infty} \mu_1^l C^{(l)}\right)\left(\sum_{l=0}^{\infty} \mu_2^l C^{(l)}\right) = \sum_{l=0}^{\infty} (\mu_1 + \mu_2)^l C^{(l)}.$$

Comparing the coefficients of $\mu_1^i \mu_2^j$ gives

$$C^{(i)}C^{(j)} = \frac{(i+j)!}{i!j!} C^{(i+j)}.$$

Since $\rho\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ we know $C^{(0)} = I$. Let $C^{(1)} = A$. We have a sequence of equations

$$C^{(1)} = A,\ C^{(2)} = \frac{1}{2}A^2,\ C^{(3)} = \frac{1}{3!}A^3,\ C^{(4)} = \frac{1}{4!}A^4, \ldots$$

but for sufficiently large l, $C^{(l)} = 0$ and $A^l = 0$. Hence A is nilpotent, and for $\mu \in \mathbf{k}$

$$\begin{aligned} \rho\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} &= 1 + \mu A + \frac{\mu^2}{2}A^2 + \frac{\mu^3}{3!}A^3 + \ldots \\ &= \exp(\mu A). \end{aligned}$$

A gives a nilpotent, and therefore singular, map $V \to V$, and so there is some non-zero $v \in V$ such that $Av = 0$. Thus

$$\rho\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}(v) = (1 + \mu A + \frac{\mu^2}{2}A^2 + \ldots)v = v$$

for any $\mu \in \mathbf{k}$. Therefore, V contains a one-dimensional submodule $\mathbf{k}v$, the trivial module. As V is irreducible $V = \mathbf{k}v$, and ρ is the unit representation.

□

7.5.3 Lemma. *When* $\mathrm{char}(\mathbf{k}) = p > 0$ *the only irreducible polynomial representation of* $\mathbf{k}^+$ *is the unit representation.*

Proof. Let V be an irreducible polynomial $\mathbf{k}^+$-module affording the representation ρ. Let $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathbf{k}$. Then

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^p = \begin{pmatrix} 1 & pa \\ 0 & 1 \end{pmatrix} = I,$$

and any eigenvalue e of $\rho\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ satisfies $e^p = 1$. So $e \in \overline{\mathbf{k}}$ and $e^p = 1$. Therefore $e^p - 1 = 0$, which implies $(e-1)^p = 0$ and hence $e = 1$. Thus all the eigenvalues of $\rho\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ are equal to 1. So there is an eigenvector $v \in V$ such that $v \neq 0$ and $\rho\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} v = v$.

Define the subspace $V' \subset V$ by

$$V' = \{v \in V; \rho\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} v = v\}.$$

We have shown $V' \neq 0$. V' is a $\mathbf{k}^+$-submodule since for any $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \in \mathbf{k}^+$, and any $v \in V'$,

$$\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} v = \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} v = \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} v,$$

and so $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} v \in V'$. Since V is irreducible, $V' = V$ and ρ is the unit representation.

□

7.5.4 Corollary. *Let $\mathbf{k}$ be any infinite field. Then the only irreducible polynomial representation of $\mathbf{k}^+$ is the unit representation.*

Proof. This follows from Lemmas 7.5.2 and 7.5.3.

□

Recall from Chapter 1 the height function defined on the set of roots Φ. In Carter [1] p.77 a total ordering is defined on the roots which is compatible with the height function, i.e. so that if $r < s$ then $h(r) \leq h(s)$. It is also shown that

$$U_{Sp}^- = \prod_{r \in \Phi^-} X_r$$

where the product is taken in increasing order.

7.5.5 Definition. Let $r \in \Phi^-$. Define $U_r^- \subset U_{sp}^-$ to be the subgroup generated by the root subgroups χ_s for all $s \leq r$ with respect to the total ordering defined by Carter.

The use of the commutator formula in Carter [1] p. 78 shows that for all $r \in \Phi^-$. $U_r^- \triangleleft U_{sp}^-$. Let $\Phi^- = \{r_1, \ldots, r_t\}$ with $r_1 < \ldots < r_t$. This provides a normal series

$$1 \subset U_{r_1} \subset \ldots \subset U_{r_t} = U_{sp}^-.$$

For $i \in \{1, \ldots, t-1\}$

$$\frac{U_{r_{i+1}}}{U_{r_i}} = \frac{U_{r_i} X_{r_{i+1}}}{U_{r_i}} \cong \frac{X_{r_{i+1}}}{U_{r_{i+1}} \cap X_{r_{i+1}}} = X_{r_{i+1}} \cong \mathbf{k}^+.$$

7.5.6 Lemma. *The only irreducible polynomial representation of U_{sp}^- is the unit representation.*

Proof. Write the above normal series as

$$1 \triangleleft U_1 \triangleleft \ldots \triangleleft U_t = U_{sp}^-,$$

where $U_i = U_{r_i}$. Let V be an irreducible polynomial U_{sp}^--module, and define a subspace $V_1 \subseteq V$ by

$$V_1 = \{v \in V; g.v = v \text{ for all } g \in U_1\}.$$

Now $U_1 = X_{r_1} \cong \mathbf{k}^+$, and since V contains an irreducible U_1-module, by Corollary 7.5.4 $V_1 \neq 0$. For all $g_1 \in U_1$, $g_2 \in U_2$ and $v \in V_1$

$$g_1(g_2.v) = g_2 g_2^{-1} g_1 g_2.v = g_2.v$$

since $g_1^{g_2} \in U_1$, and so V_1 is a non-zero U_2-module.

Define the subspace $V_2 \subseteq V_1$ by

$$V_2 = \{v \in V_1; g.v = v \text{ for all } g \in U_2\}.$$

V_1 is a U_2/U_1-module and so by Corollary 7.5.4 $V_2 \neq 0$. Let $g_2 \in U_2$, $g_3 \in U_3$ and $v \in V_2$. Then

$$g_2(g_3.v) = g_3 g_3^{-1} g_2 g_3.v = g_3.v$$

since $U_2 \triangleleft U_3$, and hence V_2 is a non-zero U_3-module .

If we continue in this way we eventually obtain a non-zero U_{sp}^--submodule $V_t \subseteq V$ given by

$$V_t = \{v \in V_{t-1}; g.v = v \text{ for all } g \in U_{sp}^-\}.$$

Since V is irreducible $V_t = V$ and V is the trivial U_{sp}^--module.

□

Recall from Section 3.2the decomposition $B^- = U_{sp}^- T$ of B^- such that $U_{sp}^- \cap T = 1$, and $U^- \triangleleft B^-$. Each element $b \in B^-$ has a unique expression $b = ut$ where $u \in U_{sp}^-$ and $t \in T$. For any $\lambda \in X(T)$ we define $\lambda : B^- \to \mathbf{k}$ called the lift of λ to B^- by putting U_{sp}^- in the kernel. We denote the lift of λ by λ as well, so $\lambda(ut) = \lambda(t)$ for all $u \in U_{sp}^-$ and $t \in T$.

7.5.7 Definition. Let D be the left Sp-module given by

$$D = Ind_{U_{sp}^-}^{sp} k_{U_{sp}^-} = \{\text{polynomial } f : Sp \to \mathbf{k}; f(ug) = f(g) \text{ for all } u \in U_{sp}^-, g \in Sp\}.$$

This is the induced right Sp-module of the trivial right U_{sp}^--module $k_{U_{sp}^-}$. The left action of Sp on D is given by

$$s.f(g) = f(gs)$$

for any $s, g \in Sp$ and $f \in D$.

For any $\lambda \in X(T)$ define

$$D^{sp}_{\lambda,\mathbf{k}} = \{\text{polynomial } f : Sp \to \mathbf{k} : f(bg) = \lambda(b)f(g) \text{ for all } b \in B^-, g \in Sp\}.$$

This is a left Sp-module under the action

$$s.f(g) = f(gs)$$

for all $g, s \in Sp$ and all $f \in D^{sp}_{\lambda,\mathbf{k}}$.

7.5.8 Lemma. *The sum $\sum_{\lambda\in X(T)} D^{sp}_{\lambda,\mathbf{k}}$ is direct.*

Proof. We need to show for any $\mu \in X(T)$

$$\left(\sum_{\substack{\lambda\in X(T)\\ \lambda\neq\mu}} D^{sp}_{\lambda,\mathbf{k}}\right) \cap D^{sp}_{\mu,\mathbf{k}} = \{0\}.$$

Let $\mu \in X(T)$ and assume that there is a non-zero polynomial function

$$f \in \left(\sum_{\substack{\lambda\in X(T)\\ \lambda\neq\mu}} D^{sp}_{\lambda,\mathbf{k}}\right) \cap D^{sp}_{\mu,\mathbf{k}}.$$

So for some finite subset $\{\lambda_1, \lambda_2 \ldots, \lambda_s\} \subset X(T)$ we have

$$f = f_1 + f_2 + \ldots + f_s$$

where $f_i \in D^{sp}_{\lambda_i,\mathbf{k}}$ for all $i \in \{1, \ldots, s\}$.

For any $g \in Sp$ and $t \in T$

$$\begin{aligned}\mu(t)f(g) &= f(tg)\\ &= f_1(tg) + \ldots + f_s(tg)\\ &= \lambda_1(t)f_1(g) + \ldots + \lambda_s(t)f_s(g).\end{aligned}$$

Choose $g \in Sp$ such that $f(g) \neq 0$. Then

$$\mu(t) = \alpha_1\lambda_1(t) + \ldots \alpha_s\lambda_s(t), \qquad (*)$$

where $\alpha_i = f_i(g)/f(g)$ for all i. Let

$$t = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & 0 & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \in T$$

with $t_1, \ldots, t_n \in \mathbf{k} - \{0\}$. If $\mu = (\mu_1, \ldots, \mu_n)$ and, for all $i \in \{1, \ldots, s\}$, $\lambda_i = (\lambda_1^i, \ldots, \lambda_n^i)$ then Equation $(*)$ gives

$$t_1^{\mu_1} \ldots t_n^{\mu_n} - \alpha_1 t_1^{\lambda_1^1} \ldots t_n^{\lambda_n^1} - \ldots - \alpha_s t_1^{\lambda_1^s} \ldots t_n^{\lambda_n^s} = 0.$$

This is a $\mathbf{k}$-linear combination of distinct monomials over $\mathbf{k}$. If we show that the monomials over $\mathbf{k}$ are linearly independent, then we have a contradiction, and so the sum is direct.

Assume for some $p \in \mathbb{N}$ that monomials in p variables are linearly independent over $\mathbf{k}$. Let $t_1, \ldots, t_{p+1}$ be variables over $\mathbf{k}$, and let

$$b_1 t_1^{\beta_1^1} \ldots t_{p+1}^{\beta_{p+1}^1} + b_2 t_1^{\beta_1^2} \ldots t_{p+1}^{\beta_{p+1}^2} + \ldots + b_s t_1^{\beta_1^s} \ldots t_{p+1}^{\beta_{p+1}^s} \qquad (\star)$$

be a $\mathbf{k}$-linear combination of distinct monomials.

For a contradiction assume that $(\star)$ equals zero. Let $\beta \in \mathbb{N}$ occur as a power of t_1 in $(\star)$, and gather together all terms involving t_1^β. The coefficient of t_1^β in the left hand side of $(\star)$ is a non-zero $\mathbf{k}$-linear combination of monomials in p variables t_2, ..., t_{p+1}, and is therefore non-zero. However we showed in the proof of Lemma 6.7.12 that natural powers of one variable t over $\mathbf{k}$ are linearly independent over any algebra over $\mathbf{k}$. This contradicts the fact that equation $(\star)$ is zero. Hence monomials in $p+1$ variables are linearly independent over $\mathbf{k}$.

We have shown the assumption is true for $p = 1$ and therefore, by induction, monomials in any finite number of variables over $\mathbf{k}$ are linearly independent over $\mathbf{k}$.

□

7.5.9 Lemma.

$$D = \bigoplus_{\lambda \in X(T)} D_{\lambda,\mathbf{k}}^{sp}.$$

Proof. Let $\lambda \in X(T)$ and let $f \in D_{\lambda,\mathbf{k}}^{sp}$. For any $u \in U_{sp}^-$ and any $g \in Sp$

$$f(ug) = \lambda(u) f(g) = f(g).$$

So $f \in D$ and we have $\bigoplus_{\lambda \in X(T)} D_{\lambda,\mathbf{k}}^{sp} \subseteq D$.

Conversely, let $f \in D$. There is a right B^--action on D given by

$$f.b(g) = f(bg)$$

for all $b \in B^-$ and all $g \in Sp$. D is closed under this action since

$$\begin{aligned} f.b_1(b_2 g) &= f(b_1 b_2 g) \\ &= \lambda(b_1)\lambda(b_2)f(g) \\ &= \lambda(b_2)\lambda(b_1)f(g) \\ &= \lambda(b_2)f(b_1 g) \\ &= \lambda(b_2)f.b_1(g) \end{aligned}$$

for all $b_1, b_2 \in B^-$ and all $g \in Sp$.

By definition, the right action of U_{Sp}^- on D is trivial, and D is a B^-/U_{Sp}^--module. Under the isomorphism $T \cong B^-/U_{Sp}^-$, D is a right T-module. Hence any $f \in D$ is contained in a sum of weight spaces in D . Hence f can be expressed as a sum

$$f = f_1 + \ldots + f_s$$

for $\lambda_1, \ldots, \lambda_s \in X(T)$ such that $f_i \in (D)^{\lambda_i, \mathbf{k}}$.

Fix $i \in \{1, \ldots, s\}$. Then

$$\begin{aligned} f_i(bg) &= (f_i.b)(g) \\ &= (f_i.t)(g) \quad \text{where } b = ut \\ &= \lambda_i(t)f_i(g) \quad \text{since } f_i \in (D)^{\lambda_i, \mathbf{k}} \\ &= \lambda_i(b)f_i(g) \end{aligned}$$

by definition of the lift of λ_i to B^-.

Hence $f_i \in D^{sp}_{\lambda_i, \mathbf{k}}$ and $f \in \sum_{\lambda \in X(T)} D^{sp}_{\lambda, \mathbf{k}}$. Therefore,

$$\bigoplus_{\lambda \in X(T)} D^{sp}_{\lambda, \mathbf{k}} \subseteq D \subseteq \bigoplus_{\lambda \in X(T)} D^{sp}_{\lambda, \mathbf{k}},$$

and $\bigoplus_{\lambda \in X(T)} D^{sp}_{\lambda, \mathbf{k}} = D$.

□

7.5.10 Lemma. *If $\lambda \in X(T)$ is not dominant then $D^{sp}_{\lambda, \mathbf{k}} = 0$.*

Proof. See Jantzen [1] p. 200.

□

7.5.11 Theorem. *Every irreducible polynomial Sp-module is isomorphic to a submodule of $D^{sp}_{\lambda,\mathbf{k}}$ for some $\lambda \in X(T)^+$.*

Proof. Let M be an irreducible polynomial Sp-module. By Lemma 7.5.6 the restriction $M_{U^-_{Sp}}$ of M to the action of U^-_{Sp} must have a composition series with all factors isomorphic to the trivial U^-_{Sp}-module $k_{U^-_{Sp}}$. Hence there is a non-zero U^-_{Sp}-homomorphism $\phi : M_{U^-_{Sp}} \to K_{U^-_{Sp}}$. We can use the homomorphism ϕ to define a map $f_v : Sp \to K_{U^-_{Sp}}$ for any $v \in M$ by setting

$$f_v(g) = \phi(g.v)$$

for all $g \in Sp$. Since ϕ is a homomorphism of polynomial U^-_{sp}-modules f_v is a polynomial function.

Let $u \in U^-_{Sp}$ and let $v \in M$. For all $g \in Sp$

$$\begin{aligned} f_v(ug) &= \phi(ug.v) \\ &= u\phi(g.v) \\ &= \phi(g.v) \end{aligned}$$

since ϕ is a U^-_{Sp}-homomorphism and the action of U^-_{Sp} is trivial. Hence $f_v \in D$. Therefore we can define a map $\psi : M \to D$ by setting

$$\psi(v) = f_v$$

for all $v \in M$. Then ψ is an Sp-homomorphism since

$$\begin{aligned} \psi(v_1 + v_2)(g) &= f_{v_1+v_2}(g) \\ &= \phi(g.(v_1 + v_2)) \\ &= \phi(g.v_1) + \phi(g.v_2) \\ &= (\psi(v_1) + \psi(v_2))(g) \end{aligned}$$

for all $v_1, v_2 \in M$ and all $g \in Sp$, and

$$\begin{aligned} (s.\psi(v))(g) &= (s.f_v)(g) \\ &= f_v(gs) \\ &= \phi(gs.v) \\ &= f_{s.v}(g) \\ &= (\psi(s.v))(g) \end{aligned}$$

for all $v \in M$ and all $g, s \in Sp$.

Let $v \in M$ satisfy $\phi(v) \neq 0$. For any $g \in Sp$ define $v_g = g^{-1}v$. Then

$$\psi(v_g)(g) = f_{v_g}(g) = \psi(g.v_g) = \phi(gg^{-1}.v) = \phi(v) \neq 0.$$

Hence $\psi : M \to D$ is a non-zero homomorphism of left Sp-modules. Since M is irreducible $\mathrm{Ker}\psi = 0$, and $M \cong \mathrm{Im}\psi \subset D$. So M is isomorphic to an irreducible submodule of D. Since D is a direct sum of the modules $D^{sp}_{\lambda,\mathbf{k}}$ as λ ranges over $X(T)$, and $D^{sp}_{\lambda,\mathbf{k}} = 0$ when $\lambda \notin X(T)^+$, M must be isomorphic to a submodule of $D^{sp}_{\lambda,\mathbf{k}}$ for some $\lambda \in X(T)^+$.

□

7.5.12 Corollary. *Every irreducible polynomial Sp-module is isomorphic to* $F^{sp}_{\lambda,\mathbf{k}}$ *for a unique* $\lambda \in X(T)^+$.

Proof. By the above theorem any irreducible polynomial Sp-module M is isomorphic to an irreducible submodule of $D^{sp}_{\lambda,\mathbf{k}}$ for some $\lambda \in X(T)^+$. Since $F^{sp}_{\lambda,\mathbf{k}}$, is the unique minimal submodule of $D^{sp}_{\lambda,\mathbf{k}}$ we must have $F^{sp}_{\lambda,\mathbf{k}} \cong M$.

Whenever $\lambda_1, \lambda_2 \in X(T)^+$ satisfy $\lambda_1 \neq \lambda_2$, $F^{sp}_{\lambda_1,\mathbf{k}} \not\cong F^{sp}_{\lambda_2,\mathbf{k}}$ as Sp-modules, and so such an $F^{sp}_{\lambda,\mathbf{k}}$ must be unique.

□

So the set of $F^{sp}_{\lambda,\mathbf{k}}$ as λ ranges over $X(T)^+$ form a complete set of irreducible polynomial modules for $Sp_{2n}(\mathbf{k})$.

7.6 Alternative Approaches.

In this chapter we briefly mention some of the other approaches to the same subject, and how they relate to this thesis.

Seshadri, Musili and Lakshmibai, [1] and [2], have developed Standard Monomial Theory and they use it to obtain a basis for $V_{\lambda,\mathbf{Z}} \subset V_{\lambda,\mathbf{C}}$ which is different from ours. The basis is indexed by standard monomials which correspond to certain collections of admissible pairs (w_1, w_2), where w_1 and w_2 are in the Weyl group such that $\frac{1}{2}(w_1(\lambda) + w_2(\lambda))$ is a weight in $V_{\lambda,\mathbf{k}}$.

De Concini in [1] and [2] discusses the Schur module $D^{sp}_{\lambda,\mathbf{k}}$, where $\mathbf{k}$ is any infinite field, and gives a basis. He uses a different definition of symplectic tableaux to index the basis, but his definition is not as intuitive as that of King's.

In [2] Wetherilt defines $V^{sp}_{\lambda,\mathbf{k}}$ as the contravariant dual to $D^{sp}_{\lambda,\mathbf{k}}$. He uses the compound contractions and expansions, and shows that $V^{sp}_{\lambda,\mathbf{k}}$ is the space of traceless tensors in $V^{gl}_{\lambda,\mathbf{k}}$.

Berele in [1] studies the irreducible module for symplectic groups over $\mathbf{C}$. He obtains an irreducible $Sp_{2n}(\mathbf{C})$-module as a quotient of the Weyl module for $Gl_{2n}(\mathbf{C})$, and gives a basis indexed by the symplectic λ-tableaux of King. He uses only the simple contraction and expansion maps.

Donkin in [1] has independently obtained the same basis for $D^{sp}_{\lambda,\mathbf{k}}$ consisting of bideterminants of King's symplectic λ-tableaux, as given in Chapter 6, by a different method.

Index of Notation.

E_P	The simple expansion $E_1(T_P)$	86
$E_{P,S}$	The simple expansion $E_1(T_{P,S})$	86
X	The set of semistandard non-symplectic (1^m)-tableaux with single set $\emptyset$	86
E	The set of simple expansions E_P for all equation sets P of a fixed length	86
ω	A word, which is a certain sequence of hatted and unhatted elements	87
Ω	The set of complete words	87
X^ω	The set of $T_R \in X$ for which R satisfies the conditions of ω	87
E^ω	The set of $E_P \in E$ for which P satisfies the conditions of ω	87
N^ω	The inverse matrix to M^ω	93
X^S	The set of non-symplectic tableaux $T_{R,S}$ for R of a fixed length	94
E^S	The set of simple expansions $E_1(T_{P,S})$ for P of a fixed length	94
${}_n\Upsilon_S$	A map from $\{1,...,n\} \setminus mod(S)$ to $\{1,...,n-\varsigma\}$	95
w_t^+	An operator on column bideterminants	99
$E_{s,t}$	The expansion operator of degree t acting on column s	109
$\mathcal{T}_{sp}$	The set of all symplectic λ-tableaux with entries from $I \cup \overline{I}$	111
$\Gamma_{i,j}$	A subset of $\mathcal{T}_{sp}$	111
$\Gamma_{i,\bar{j}}$	A subset of $\mathcal{T}_{sp}$	111
$\Gamma_{i,\bar{i}}$	A subset of $\mathcal{T}_{sp}$	111
n_0	A certain element in the normalizer of the Weyl group	123
$\mathcal{T}_{col}$	The set of $T \in \mathcal{T}$ with columns strictly increasing w.r.t. to the usual ordering	127
$\mathcal{T}_{\underline{col}}$	The set of $T \in \mathcal{T}$ with columns strictly increasing w.r.t. another ordering	127
$\mathcal{T}_{ss}$	The set of semistandard λ-tableaux in $\mathcal{T}$	131
$\mathcal{T}_{\underline{sp}}$	The set of symplectic λ-tableaux in $\mathcal{T}$ w.r.t. an unusual ordering	131

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